Lesson 2

Read Chapter 1

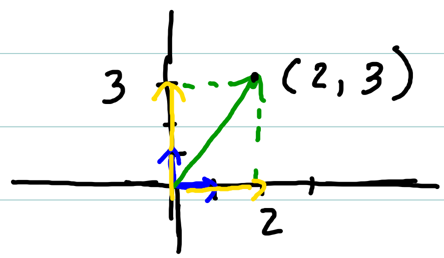
Change of bases
Diagonalization
Similar matrices

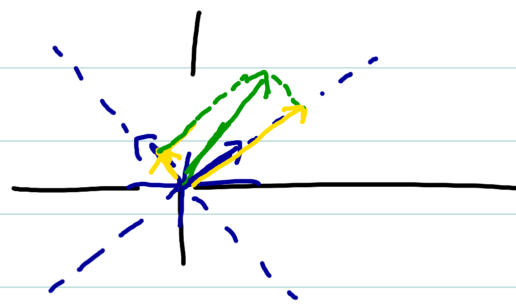
From last time:

$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\quad\ e_1 \quad\ e_2 \qquad\qquad\qquad b_1 \quad\ b_2$

are two bases in $\mathbb{R}^2$



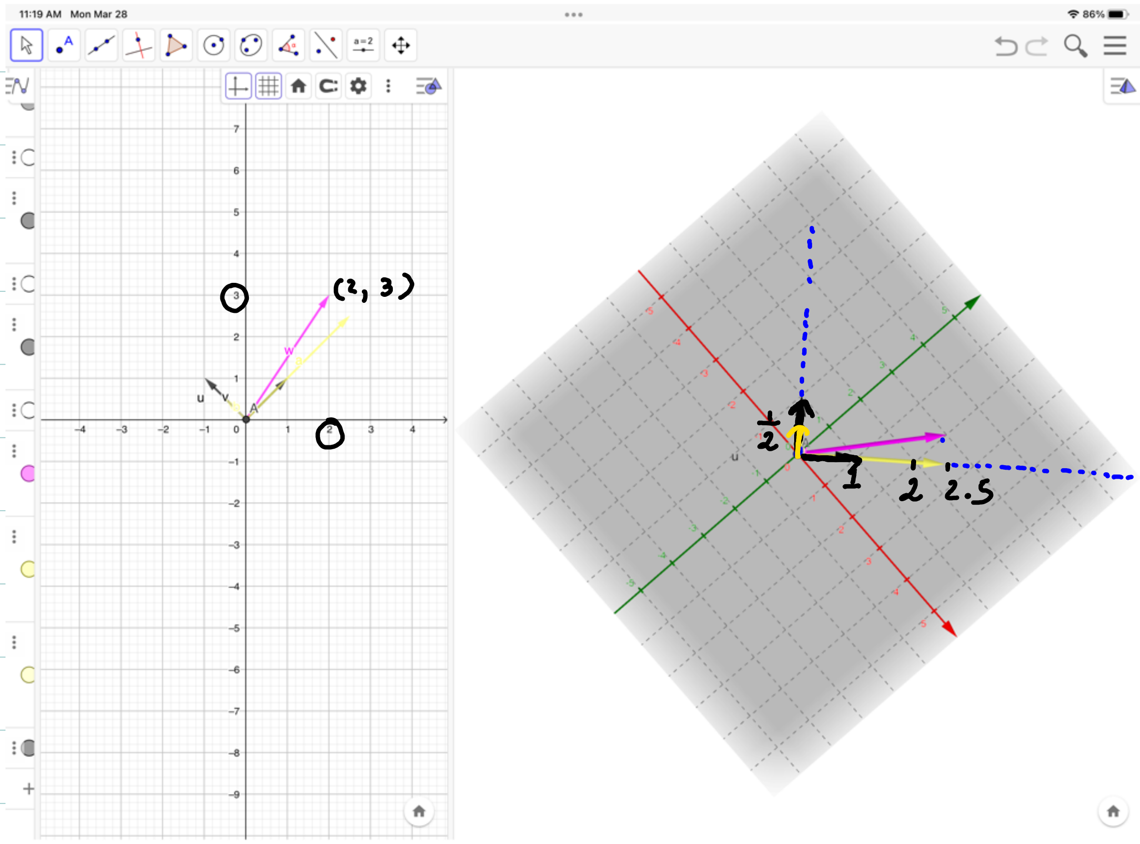


$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 e_1 + 3 e_2$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2} b_1 + \frac{1}{2} b_2$

$\left[\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right]_B = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$

coordinates of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ wrt B


(2, 3)
1/2
1
2
2.5

Def: the canonical basis for $\mathbb{R}^n$ is E: $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ .... $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Change of basis matrices:

Theorem:

Given: $E: e_1 \dots e_n$ $\quad B: b_1, \dots b_n$ basis for $\mathbb{R}^n$

There are matrices U_E^B, U_B^E s.t

$$U_E^B v = [v]_B$$

$$U_B^E [v]_B = v$$

Matrices of change of basis

for all v in $\mathbb{R}^n$.

$$U_B^E = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}, \quad U_E^B = (U_B^E)^{-1}$$

Example

If $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$U_B^E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad U_E^B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$

$U_E^B = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$ (inverse calculation)

Check that:

$U_E^B \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$ i.e. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$U_B^E \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Proof of theorem if $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

First we show $\forall v \in \mathbb{R}^2$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = v$:

why? $[v]_B = \begin{bmatrix} h \\ k \end{bmatrix}$ means $v = h \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h-k \\ h+k \end{bmatrix} = h \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} -1 \\ 1 \end{bmatrix} = v$$

Then we show $\forall v \in \mathbb{R}^2$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v = [v]_B$:

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is invertible why?

From $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = v$ get

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v$$

$$[v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v$$

Def $A \in \mathbb{R}^{n\times n}$ is diagonalizable if we can write
$A = P D P^{-1}$ for some P, and diagonal D

Th: $A \in \mathbb{R}^{n\times n}$ is diagonalizable iff there
is a basis for $\mathbb{R}^n$ made of eigenvectors
of A, that is if $\lambda_1 \cdots \lambda_k$ are the
distinct eigenvalues of A and
E_{λ_1} has basis B_1, E_{λ_2} has basis B_2
.... E_{λ_k} has basis B_k
then $B = B_1 \cup B_2 \cdots \cup B_k$ is a basis for $\mathbb{R}^n$

Ex $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable (in $\mathbb{R}$)
why?

no eigenvalues

Ex $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable why?

Only eigenvalue $\lambda = 1$
$E_1 = \text{Null}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ which has dimension 1

$p(\lambda) = (\lambda - 1)^2$ 1 has algebraic multiplicity
2 and geometric multiplicity 1

proof of th: first suppose A is diagonalizable i.e $A = PDP^{-1}$ for some P and diagonal D. since P is invertible, the columns of P are a basis for R.

$$P = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \qquad B = b_1 \cdots b_n$$

then $P = U_B^E \qquad P^{-1} = U_E^B$

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$ We want to show

$b_1, b_2, \dots b_n$ are eigenvectors of A for eigenvalues $d_1, d_2, \dots d_n$:

$$A b_1 = P D P^{-1} b_1 = P D U_E^B b_1 = P D [b_1]_B$$
$$= P D \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 P \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 b_1$$

so b_1 is an eigenvector for A with eigenvalue d_1

Similarly $Ab_2 = d_2 b_2$, $Ab_3 = d_3 b_3$ .....

Viceversa suppose $A \in R^{n\times n}$ and $B = b_1 \cdots b_n$ is a basis for R^n consisting of eigenvectors of A for eigenvalues $\lambda_1 \cdots \lambda_n$ (not necessarily distinct). We want to show A is diagonalizable.

Take $P = \begin{bmatrix} b_1 \cdots b_n \end{bmatrix}$ $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Then $A b_i = \lambda_i b_i$

$P D P^{-1} b_i = P D U_{\mathcal{E}}^{B} b_i = P D [b_i]_B = P D e_i$

$= P D \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = P \lambda_i e_i = \lambda_i P e_i = \lambda_i b_i$

so $A b_i = P D P^{-1} b_i$ for $i = 1 \cdots n$

Is it enough to conclude $A = PDP^{-1}$?

yes $e_1 = k_1 b_1 + k_2 b_2 + \cdots + k_n b_n$

$A e_1 = A(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n) = k_1 (A b_1) + k_2 (A b_2) + \cdots + k_n (A b_n)$ = first column of A

$PDP^{-1} e_1 = PDP^{-1}(k_1 b_1 + k_2 b_2 + \cdots + k_n b_n) =$
$k_1 (PDP^{-1} b_1) + k_2 (PDP^{-1} b_2) + \cdots + k_n (PDP^{-1} b_n)$
= first column of PDP^{-1}

Then first column of A = first column of PDP^{-1}. Similarly we can get that all columns are equal

Def: $A, B \in R^{n \times n}$ are similar if $A = PBP^{-1}$ for some P

we will write $A \sim B$

Note if $A = PBP^{-1}$ then $P^{-1}AP = B$ so $B = QAQ^{-1}$ where $Q = P^{-1}$

th: similar matrices have the same eigenvalues, characteristic polynomial

Proof: Assume $A = P B P^{-1}$ then

$p_A(\lambda) = \det(A - \lambda I) = \det(PBP^{-1} - \lambda I) =$

$\det(PBP^{-1} - P\lambda I P^{-1}) =$

$\det(P(B - \lambda I)P^{-1}) =$

$\det(P)\det(B-\lambda I)\det(P^{-1}) =$

$\det(B - \lambda I) =$ since $\det(P^{-1}) = \dfrac{1}{\det(P)}$

$= p_B(\lambda)$

what about eigenvectors? In general not the same.