

Lesson 2

Read Chapter 1

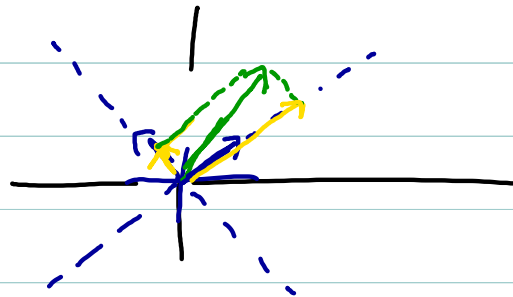
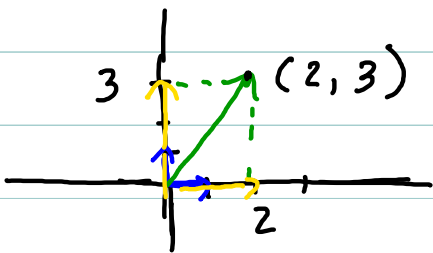
Change of bases
Diagonalization
Similar matrices

From last time:

$$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

e_1 e_2 b_1 b_2

ere two bases in \mathbb{R}^2



$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2e_1 + 3e_2$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2}b_1 + \frac{1}{2}b_2$$

$$\left[\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right]_B = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$$

coordinates of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ wrt B

The image displays a digital workspace with two main graphical areas. On the left is a standard Cartesian coordinate system with x and y axes ranging from -4 to 4. A point is plotted at (2, 3) and labeled with a circled '3' and the coordinates (2, 3). Two vectors, u and v , originate from the origin (0, 0). Vector u is yellow and points to (1, 1), while vector v is pink and points to (2, 3). On the right is a rotated grid, tilted 45 degrees counter-clockwise. A point is plotted at (2, 3) in the rotated system, with a circled '3' and the coordinates (2, 3). Two vectors, u and v , originate from the origin of the rotated grid. Vector u is yellow and points to (1, 1) in the rotated system, with a label $\frac{1}{2}$ below it. Vector v is pink and points to (2, 3) in the rotated system, with labels 1, 2, and 2.5 below it. The rotated grid has red and green axes. The workspace includes a toolbar at the top with various drawing tools and a home button at the bottom right.

Def: the canonical basis for \mathbb{R}^n is $E: e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$
 $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \dots e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Change of basis matrices:

Theorem:

Given: $E: e_1, \dots, e_n$ $B: b_1, \dots, b_n$

There are matrices U_E^B, U_B^E s.t. basis for \mathbb{R}^n

$$U_E^B v = [v]_B$$

Matrices of change
of basis

$$U_B^E [v]_B = v$$

for all v in \mathbb{R}^n .

$$U_B^E = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}, \quad U_E^B = (U_B^E)^{-1}$$

Example

$$\text{If } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$U_B^E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad U_E^B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$U_E^B = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \quad (\text{inverse calculation})$$

Check that :

$$U_E^B \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$U_B^E \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Proof of theorem if $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

First we show $\forall v \in \mathbb{R}^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = v$:

why? $[v]_B = \begin{bmatrix} h \\ k \end{bmatrix}$ means $v = h \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h - k \\ h + k \end{bmatrix} = h \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} -1 \\ 1 \end{bmatrix} = v$$

Then we show $\forall v \in \mathbb{R}^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v = [v]_B$:

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is invertible why?

From $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = v$ get

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v$$

$$[v]_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} v$$

Def $A \in \mathbb{R}^{n \times n}$ is diagonalizable if we can write
 $A = P D P^{-1}$ for some P , and diagonal D

Th: $A \in \mathbb{R}^{n \times n}$ is diagonalizable iff there
is a basis for \mathbb{R}^n made of eigenvectors
of A , that is if d_1, \dots, d_k are the
distinct eigenvalues of A and
 E_{d_1} has basis B_1 , E_{d_2} has basis B_2
..... E_{d_k} has basis B_k
then $B = B_1 \cup B_2 \dots \cup B_k$ is a basis for \mathbb{R}^n

Ex $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is not diagonalizable (in \mathbb{R})
why?

no eigenvalues

Ex $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable why?

only eigenvalue $d=1$

$E_1 = \text{Null} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ which has dimension 1

$p(d) = (d-1)^2$ 1 has algebraic multiplicity
2 and geometric multiplicity 1

proof of th: first suppose A is diagonalizable i.e. $A = P D P^{-1}$ for some P and diagonal D . since P is invertible, the columns of P are a basis for \mathbb{R}^n .

$$P = [b_1 \cdots b_n] \quad B = b_1 \cdots b_n$$

then $P = U_B^E \quad P^{-1} = U_E^B$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{We want to show}$$

b_1, b_2, \dots, b_n are eigenvectors of A for eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$:

$$\begin{aligned} A b_1 &= P D P^{-1} b_1 = P D U_E^B b_1 = P D [b_1]_B \\ &= P D \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 P \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 b_1 \end{aligned}$$

so b_1 is an eigenvector for A with eigenvalue λ_1 ,
Similarly $A b_2 = \lambda_2 b_2, A b_3 = \lambda_3 b_3 \dots$

Viceversa suppose $A \in \mathbb{R}^{n \times n}$ and $B = b_1, \dots, b_n$ is a basis for \mathbb{R}^n consisting of eigenvectors of A for eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). We want to show A is diagonalizable.

$$\text{Take } P = [b_1 \dots b_n] \quad D = \begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix}$$

$$\begin{aligned} \text{Then } A b_l &= \lambda_l b_l \\ P D P^{-1} b_l &= P D U_{\mathcal{B}}^{\mathcal{B}} b_l = P D [b_l]_{\mathcal{B}} = P D e_l \\ &= P D \begin{bmatrix} 0 \\ \vdots \\ \lambda_l \\ \vdots \\ 0 \end{bmatrix} = P \lambda_l e_l = \lambda_l P e_l = \lambda_l b_l \end{aligned}$$

$$\text{so } A b_l = P D P^{-1} b_l \quad \text{for } l=1 \dots n$$

Is it enough to conclude $A = P D P^{-1}$?

$$\text{yes } e_1 = k_1 b_1 + k_2 b_2 + \dots + k_n b_n$$

$$\begin{aligned} A e_1 &= A (k_1 b_1 + k_2 b_2 + \dots + k_n b_n) = k_1 (A b_1) + k_2 (A b_2) + \\ &+ \dots + k_n (A b_n) = \text{first column of } A \end{aligned}$$

$$\begin{aligned} P D P^{-1} e_1 &= P D P^{-1} (k_1 b_1 + k_2 b_2 + \dots + k_n b_n) = \\ k_1 (P D P^{-1} b_1) &+ k_2 (P D P^{-1} b_2) + \dots + k_n (P D P^{-1} b_n) \\ &= \text{first column of } P D P^{-1} \end{aligned}$$

Then first column of $A =$ first column of $P D P^{-1}$. Similarly we can get that all columns are equal

Def: $A, B \in \mathbb{R}^{n \times n}$ are similar
if $A = P B P^{-1}$ for some P

We will write $A \sim B$

Note if $A = P B P^{-1}$ then
 $P^{-1} A P = B$ so $B = Q A Q^{-1}$ where $Q = P^{-1}$

Th: similar matrices have the
same eigenvalues, characteristic
polynomial

Proof: Assume $A = P B P^{-1}$ then

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) = \det(P B P^{-1} - \lambda I) = \\ &= \det(P B P^{-1} - P \lambda I P^{-1}) = \\ &= \det(P (B - \lambda I) P^{-1}) = \\ &= \det(P) \det(B - \lambda I) \det(P^{-1}) = \\ &= \det(B - \lambda I) = p_B(\lambda) \end{aligned}$$

since $\det(P^{-1}) = \frac{1}{\det(P)}$

What about eigenvectors? In general not
the same.