Lesson 18

Read ch 7

Help with hw 7 #9

polynomial spaces

Recall :

Given a (connected) graph $G = \langle V, E \rangle$ with n vertices

- A cut is a partition A, $V-A$ of V

- Density of a cut $\varphi_G(A) = n \dfrac{|E(A, V-A)|}{|A||V-A|}$

- Density of G is $\varphi_G = \min\limits_{\substack{A \subseteq V \\ A \neq \emptyset \\ A \neq V}} \varphi_G(A)$

In HW 6 # 4 you need to prove $\lambda_2 \leq \varphi_G$

λ_2 is the second smallest eigenvalue of L_G

Goal:

$$\lambda_2 \le \varphi_G = \min_{A \subseteq V} \varphi_G(A, V-A) = \min_{A \subseteq V} n \frac{|E(A, V-A)|}{|A|\,|V-A|}$$

λ_2 = second eigenvalue of L_G

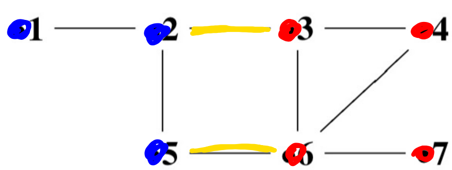
Idea: work with vectors and quadratic forms

Def For $A \subseteq V$ $\quad c_A = (c_1, \ldots, c_n)^T$

characteristic vector of A if

$c_i = 1$ if $i \in A$ $\qquad c_i = 0$ if $i \notin A$

Example



$$A = \{1, 2, 5\}$$

$$c_A = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_A^T L_G c_A = \sum_{(i,j) \in E} \left((c_A)_i - (c_A)_j \right)^2$$

$$= 2 = |E(A, V-A)|$$

Th : For any graph $G = \langle V, E \rangle$

for any $A \subseteq V$ we have :

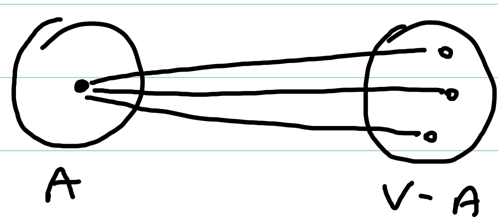
$$c_A^T L_G c_A = \sum_{\{i,j\} \in E} (c_i - c_j)^2 = |E(A, V-A)|$$

Th: For any graph $G = \langle V, E \rangle$ for any $A \subseteq V$

$$|A|\,|V-A| = C_A^T L_{K_n} C_A$$

where K_n is the complete graph on n vertices

proof: $C_A^T L_{K_n} C_A$ = number of edges from A to $V-A$ in K_n



This number is $|A| \cdot |V-A|$

$$\varphi_G = \min_{A} \; n \frac{|E(A, V-A)|}{|A||V-A|} =$$

$$= \min_{C_A} \; n \frac{C_A^T L_G C_A}{C_A^T L_{K_n} C_A}$$

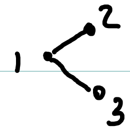
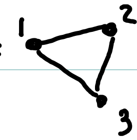
Def: $Q(x) = n \dfrac{x^T L_G x}{x^T L_{K_n} x} = n \dfrac{\sum_{\{i,j\} \in E} (x_i - x_j)^2}{\sum_{\substack{\{i,j\} \in K_n \\ \| \\ 1 \le i < j \le n}} (x_i - x_j)^2}$

n is the number of vertices in G

$$\varphi_G = \min_{\substack{x \text{ is a } 0/1 \text{ vector} \\ x \neq 0 \quad x \neq \mathbb{1}}} Q(x)$$

Note $Q(x)$ if $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a rational functi
i.e the quotient of two polynomials

$Q(C_A)$ is a number

Example $G =$  $K_3 =$ 

$$L_G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad L_{K_3} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$Q(x) = \frac{3 \; [x_1\, x_2\, x_3]\, L_G \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}{[x_1\, x_2\, x_3]\, L_{K_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = 3\,\frac{(x_1-x_2)^2 + (x_1-x_3)^2}{(x_1-x_2)^2 + (x_1-x_3)^2 + (x_2-x_3)^2}$$

$V = \{1, 2, 3\}$ if $A = \{1\}$ $c_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Q(c_A) = 3\,\frac{[1\,0\,0]\, L_G \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{[1\,0\,0]\, L_{K_3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = 3 \cdot \frac{2}{2} = 3$$

Hw 6 #4

To make the minimum easier to calculate, change (increase) the vectors we consider

Let $\mu = \min\limits_{\substack{x \in \mathbb{R}^n \\ x \neq k\mathbb{1}}} Q(x)$

Then $\mu \leq \varphi_G$

Hw goal : $d_2 = \mu$

$$\mu = \min_{\substack{x \in \mathbb{R}^n \\ x \neq k\mathbb{1}}} n \, \frac{x^T L_G \, x}{x^T L_{K_n} x} \qquad \left(= Q(x) \right)$$

(A) prove $\mu = \min\limits_{\substack{x \perp \mathbb{1} \\ \|x\| = 1}} x^T L_G \, x$

i) $Q(x) = Q(x + t\mathbb{1})$

ii) $Q(x) = Q(\alpha x)$ $\qquad Q(x) = Q(e)$

iii) If $x \perp \mathbb{1}$ $\|x\| = 1$ $\qquad x^T L_{K_n} x = n$

(B) Prove $\lambda_2 = \mu$

Recall that L_G is symmetric and eigenvectors of a symmetric matrix are orthogonal

Eigenvalues for L_G are $0, \lambda_2, \lambda_3 \ldots\ldots \lambda_n$ with eigenvectors $\mathbb{1}$ v_2 $v_3 \ldots\ldots v_n$ we can always choose them to be orthogonal with $\|v_i\| = 1$

Chapter 7

Vectors : objects that can be added and multiplied by scalars ("numbers") s.t algebra of these operations follows "nice rules"

Vector spaces other than R^n

$R^{m \times n}$ $\qquad R[x]_{\leq d}$ $\qquad R[x]$

$R[x]$ set of all polynomials in one variable with real coefficients.

$R[x]_{\leq d} \subseteq R[x]$ all polynomials in $R[x]$ with degree $\leq d$

Ex $2x^5 - x^4 + 5x + 3$ is in $R[x]$ and in $R_{\leq 5}[x]$ or $R_{\leq 10}[x]$

1) We can add polynomials :

$$(x^5-3x+1) + (x^4-x^2-5x+7) = x^5+x^4-x^2-8x+8$$

2) We can multiply a polynomial by a scalar

$$3(x^5-3x+1) = 3x^5-9x+3$$

Algebra is nice, so polynomials are vectors.

So we can ask "vector questions":

Ex: are x^5-3x+1, x^4-x^2-5x+7 linearly independent ?

Answer :

Suppose $k(x^5-3x+1)+h(x^4-x^2-5x+7) = 0$ (0 polynomial)

then $kx^5+hx^4-hx^2-(3x+5h)x+k+7h = 0$

so $k=0$ $h=0$ yes linearly indep

Note let's view these polynomials as polynomials in $\mathbb{R}[x]_{\le 5}$

$$x^5-3x+1 \rightsquigarrow \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad x^4-x^2-5x+7 \rightsquigarrow \begin{bmatrix} 7 \\ -5 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

These two vectors are linearly independent

Idea :

$$\mathbb{R}[x]_{\le d} \approx \mathbb{R}^{d+1}$$

identify $a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$

with

$$[a_0, a_1, \dots, a_{d-1}, a_d]^T$$

Answer linear algebra questions about polynomials in $\mathbb{R}[x]_{\le d}$ by answering questions about vectors in $\mathbb{R}^{d+1}$

What is a basis for $\mathbb{R}[x]_{\le d}$?

What is a basis for $\mathbb{R}^{d+1}$?

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad \dots \qquad \downarrow$$

$$1 \qquad x \qquad \dots \qquad x^d$$

Would this idea work for $\mathbb{R}[x]$?

$\mathbb{R}[x] \overset{?}{\approx} \mathbb{R}$

What is a basis for $\mathbb{R}[x]$?

Th $\mathbb{R}[x]$ does not have a finite basis

Proof : assume by contradiction $p_1 \cdots p_n$ is a basis for $\mathbb{R}[x]$; let $q(x)$ be a polynomial of degree $>$ deg p_i $i = 1 \cdots n$ then there are scalars $h_1, h_2 \dots h_n$ with

$$q(x) = h_1 p_1 + h_2 p_2 + \cdots + h_n p_n$$

but $q(x)$ has degree $\leq \max(\deg p_i)$: contradiction

A basis for $\mathbb{R}[x]$ is $1 \; x \; x^2 \cdots x^n \cdots\cdots$

it has infinitely many elements.

Lineer Transformations :

$$T: \mathbb{R}[x]_{\le 1} \to \mathbb{R}[x]_{\le 2}$$
$$T(p) = \int_0^x p(t)\,dt$$

Is a linear transformation

Check :

1) Is $T(kv) = kT(v)$?

$$\int_0^x kp(t)\,dt = k\int_0^x p(t)\,dt$$ yes from calculus

2) Is $T(v+w) = T(v) + T(w)$?

$$\int_0^x (p(t)+q(t))\,dt = \int_0^x p(t)\,dt + \int_0^x q(t)\,dt$$ yes from calculus

What is the matrix of T ?

Answer 1

$$\begin{tikzcd}
T \quad \mathbb{R}[x]_{\le 1} \arrow[r] & \mathbb{R}[x]_{\le 2} \\
p = ax+b \arrow[r] \arrow[d] & q = cx^2+dx+e \arrow[d] \\
S \quad \mathbb{R}^2 \arrow[r] & \mathbb{R}^3 \\
\begin{bmatrix} a \\ b \end{bmatrix} & \begin{bmatrix} c \\ d \\ e \end{bmatrix}
\end{tikzcd}$$

$$M \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \\ e \end{bmatrix}$$

$$M = \begin{bmatrix} S\begin{bmatrix} 1 \\ 0 \end{bmatrix} & S\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$1 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad T(1) = x \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$x \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad T(x) = \frac{x^2}{2} \to \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 1/2\, a \end{bmatrix}$$

tells us $\displaystyle\int_0^x b + at\,dt = bx + \frac{1}{2}ax^2$

Answer 2: Fix bases $B_1 = 1, x$ in $\mathbb{R}[x]_{\le 1}$ and $B_2 = 1, x, x^2$ in $\mathbb{R}[x]_{\le 2}$ then we want M st.

$$T: \mathbb{R}[x]_{\le 1} \to \mathbb{R}[x]_{\le 2}$$

$$M[p]_{B_1} = [T(p)]_{B_2}$$

$ax + b = b \cdot 1 + a \cdot x$ so $[ax+b]_{B_1} = \begin{bmatrix} b \\ a \end{bmatrix}$

$cx^2 + dx + e = e \cdot 1 + d \cdot x + c \cdot x^2$ $[cx^2 + dx + e]_{B_2} = \begin{bmatrix} e \\ d \\ c \end{bmatrix}$

To find M:

$$M[1]_{B_1} = \overbrace{M\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\text{first column of } M} = [T(1)]_{B_2}$$

$$M[x]_{B_1} = \underbrace{M\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{second column of } M} = [T(x)]_{B_2}$$

so $M = \begin{bmatrix} [T(1)]_{B_2} & [T(x)]_{B_2} \end{bmatrix}$

$T(1) = x$ $\quad T(x) = \frac{1}{2}x^2$

$[x]_{B_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \left[\frac{1}{2}x^2\right]_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$