Lesson 16

Read ch 6

Adjacency matrix
Laplacian

Ex Is $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ PSD ; Is it PD ?

1) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4xy + 4y^2 = (x+2y)^2 \geq 0$

but for example for $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $x^2 + 4xy + 4y^2 = 0$

so PSD but not PD.

Ru

2) Eigenvalues are so tions of $(2-x)(4-x)-4 =$

$x^2 - 5x = 0$ $\lambda = 0, 5$ ≥ 0

3) From proof of th

Write $A = P \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} P'$

$B^T = P \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{5} \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{5} \end{bmatrix} P^T$

B is 2×2 det $B = 0$ so columns of P are not linearly indep

3) From typed notes

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has rank 1 $= \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{B}$

Note: Columns of B not linearly independent

This B is different from the B we could construct following proof from previous lesson

Using SVD we will learn how to write any rank 1 matrix $A = \sigma_1 u_1^T v_1$

Rank 1 PSD can be written as $v v^T$ with $v \neq 0$

4) Principal minors : $1, 4, 0$

(leading principal minors $1, 0$)

Ex Is $A \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ PSD? PD?

1) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 - 4xy + 5y^2$

$= 2(x-y)^2 + 3y^2 > 0$ if $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Trick: complete square

2) Eigenvalues 1, 6

3) $A = \begin{bmatrix} 2/\sqrt5 & -1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 2/\sqrt5 & 1/\sqrt5 \\ 1/\sqrt5 & 2/\sqrt5 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt6 \end{bmatrix} \begin{bmatrix} 2/\sqrt5 & 1/\sqrt5 \\ -1/\sqrt5 & 2/\sqrt5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt5 & 1/\sqrt5 \\ -1/\sqrt{30} & 2/\sqrt{30} \end{bmatrix}$

$A = B^T \cdot B$

columns of B are linearly independent.

4) Leading principal minors : 2, $10 - (-2)(-2) = 6$

Recall how to find local max/min of

$f(x) = x^2 + 4x$

① Find critical value $\quad f'(x) = 2x + 4 = 0 \qquad x = -2$

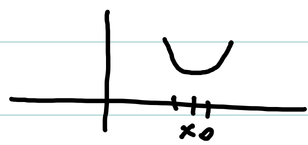
② Classify critical value. Second derivative test:

$f''(x) = 2 \qquad f''(-2) = 2 > 0$, f has local min at $x = -2$

why ? Taylor Series :

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \text{other stuff}$$

very small if x is close to x_0



The Second Derivative Test for Functions of Two Variables

1. Find all critical points of $f(x, y)$.

2. Compute

$$D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})(f_{yx}),$$

   and evaluate it at each critical point.

   a. If $D > 0$, then f has a local max or min at the critical point. To see which, look at the sign of f_{xx}:
      - If $f_{xx} > 0$, then f has a local minimum at the critical point.
      - If $f_{xx} < 0$, then f has a local maximum at the critical point.

   b. If $D < 0$ then f has a saddle point at the critical point.

   c. If $D = 0$, there could be a local max, local min, or neither (i.e., the test in inconclusive).

What about functions of 2 variables? $z = f(x\ y)$

Taylor series

$$f(x\ y) = f(x_0, y_0) + \nabla f(x_0\ y_0)^T \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \frac{1}{2} \underbrace{[x-x_0 \quad y-y_0]\, H(x_0\ y_0) \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}}_{\text{quadratic form}}$$

$+$ other stuff which is small when $\begin{bmatrix} x \\ y \end{bmatrix}$ is close to $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \qquad H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

usually symmetric $f_{xy} = f_{yx}$

critical value $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ if $\nabla f(x_0\ y_0) = \begin{bmatrix} f_x(x_0\ y_0) \\ f_y(x_0\ y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is local min at $(x_0\ y_0)$ if $H(x_0\ y_0)$ is PD i.e

$f_{xx}(x_0\ y_0) > 0 \qquad f_{xx}(x_0\ y_0)\, f_{yy}(x_0\ y_0) - f_{xy}(x_0\ y_0)\, f_{yx}(x_0 y_0) > 0$

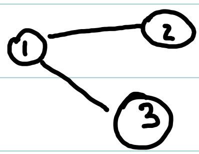
(condition 4: leading principal minors > 0)

There is a notion of negative definite matrix that explains the "local max" part of the test you find in a multivariable calculus book.

Graph Connectivity

We will consider undirected graphs.

An undirected Graph G has vertices $1 \dots n$ for some $n \in \mathbb{N}$, and edges $\{i,j\}$



$G = \langle V = \{1,2,3\} \quad E = \{\{1,2\}, \{2,3\}\} \rangle$

Def For a graph $G = \langle V, E \rangle$ and any vertex $i \in G$. The degree of i is the number of edges in E that touch i.

For G above

$\deg(1) = 2$

$\deg(2) = 1$

$\deg(3) = 1$

Adjacency matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ symmetric

PSD ? PD ?

Not PD since one of the minors is 0

$x^T A x = 2x_1x_2 + 2x_1x_3$ if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$x^T A x = -4$ not PSD

Def: Given a graph $G = \langle \{1 \cdots n\}, E \rangle$

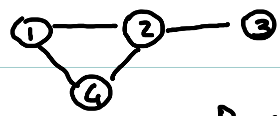
D_G is the diagonal $n \times n$ matrix $\begin{bmatrix} deg(1) & 0 \cdots & 0 \\ 0 & deg(2) \cdots & 0 \\ 0 & \cdots & deg(n) \end{bmatrix}$

A_G is the adjacency matrix of G: $a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{ow} \end{cases}$

$L_G = D_G - A_G$ is the Laplacian of G

Spectral graph theory studies properties of graphs by studying the spectrum (eigenvalues eigenvectors) of L_G

Example



$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad D_G \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$D_G - A_G = L_G = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

what do the rows and columns of L_G sum to?

L_G is symmetric, is it PSD ? is it PD ?

A_G is symmetric, could it be PSD, PD ?

No $\text{trace}(A_G) = 0 = \lambda_1 + \cdots + \lambda_n$ so some eigenvalues of A_G are positive other must be negative.

Note :

$\mathbb{1}^t L_G = 0$ and $L_G \mathbb{1} = 0$

So 0 is an eigenvalue for L_G $\quad$ L_G not PD

$\mathbb{1}$ is an eigenvector for $\lambda = 0$.

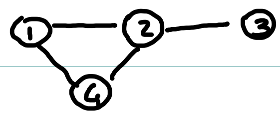
but we will show L is PSD

Def: Given $n \in \mathbb{N}$ define

G_{ij} the $n \times n$ matrix with $a_{ii} = a_{jj} = 1$

$a_{ij} = a_{ji} = -1$ all other entries $= 0$

$$G_{ij} = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & i & & & j & & & \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & & & 1 & 0 \cdots 0 & -1 & \cdots & 0 \\ \vdots & & & 0 & & \vdots & & \vdots \\ 0 & & & -1 & 0 \cdots 0 & 1 & & \vdots \\ \vdots & & & \vdots & & \vdots & & \vdots \\ 0 & & & 0 & & 0 & \cdots & 0 \end{bmatrix} \end{array}$$



$$L_G = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Let's make G_{ij} for any edge $\{i,j\}$ in G $\quad n=4$

$$G_{12} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_{14} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad G_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad G_{12} + G_{14} + G_{23} + G_{24} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix} = L_G$$

Th $L_G = \sum_{\{i,j\} \in E} G_{ij}$

Proof:

Consider $M = \sum_{\{i,j\} \in E} G_{ij}$ $\quad M = (m_{ij})$

Look at position ii for $1 \leq i \leq n$:
for any edge between vertex i and some other vertex k there is a matrix G_{ik} with 1 in position ii so $m_{ii} = \deg(i)$
Look at position ij: there is a matrix G_{ij} with -1 in position ij iff there is an edge in G between vertices i and j, therefore $(m_{ij}) = \begin{cases} 0 & \text{if } \{i\ j\} \text{ is not in } E \\ -1 & \text{if } \{i\ j\} \text{ is in } E \end{cases}$

Therefore $M = L_G$

Th: L_G is PSD

Proof $[x_1 \cdots\cdots x_n] \underbrace{\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} \vdots & \vdots \\ 1 & -1 \\ -1 & 1 \\ \vdots & \vdots \end{bmatrix} \end{array}}_{G_{ij}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_i^2 + x_j^2 - 2x_i x_j$

$= (x_i - x_j)^2$

$$x^T L_G x = x^T \Big(\sum_{\{i,j\} \in E} G_{ij} \Big) x = \sum_{\{i,j\} \in E} (x_i - x_j)^2 \geq 0$$

Note:

Typed notes talk about the directed node-edge incidence matrix, I have not talked about this matrix in class.

Given an undirected graph G with n vertices, L_G is PSD (but not PD): (recall $\mathbb{1}$ is an eigenvector for $\lambda = 0$)

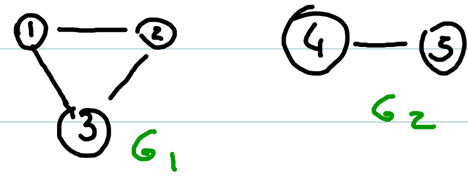
it has eigenvalues $\lambda_1 = 0 \le \lambda_2 \le \cdots \le \lambda_n$

Th: G is connected (There is a path between any 2 vertices) iff $\lambda_2 > 0$ (i.e $\lambda = 0$ has multiplicity 1)

connected

not connected

I will not give a general proof in class, but the discussion of the next example should give you an idea of how to prove this theorem.

Example 6:  Not connected

$$L_G = \left[\begin{array}{ccc|cc} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}$$

$\lambda_1 = 0$ is an eigenvalue for L_G

what is the multiplicity of $\lambda_1 = 0$?

Two linearly independent eigenvectors for $\lambda = 0$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_3 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{1}_2 \end{bmatrix}$

therefore 0 has multiplicity at least 2.

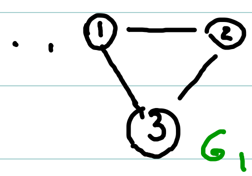
Does $\lambda_1 = 0$ have multiplicity > 2?

suppose $u \neq 0$ $\quad L_G u = \vec{0}$

$$\begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix}\right\} = w_1 \\ \left.\begin{matrix} u_4 \\ u_5 \end{matrix}\right\} = w_2 \end{bmatrix} = \begin{bmatrix} L_{G_1} w_1 \\ L_{G_2} w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So $\quad L_{G_1} w_1 = 0 \quad L_{G_2} w_2 = 0$

what can w_1 look like? Recall G_1 and G_2 are connected.



G_1

$$L_{G_1} w_1 = \vec{0}$$

Th: If G is a connected graph then for L_G, $E_0 = \text{span}(\mathbb{1})$
i.e. 0 has multiplicity 1.

Proof: assume $L_G w = 0$

Then $w^T L_G w = 0 = \sum_{\{i,j\} \in E} (w_i - w_j)^2$

so if there is an edge from i to j in G

$w_i = w_j$. What can w look like?

$$w = \begin{bmatrix} \vdots \\ w_h \\ \vdots \\ w_k \\ \vdots \end{bmatrix} \begin{matrix} \\ \longrightarrow \text{position } h \\ \\ \longrightarrow \text{position } k \\ \\ \end{matrix}$$

Since G is connected,

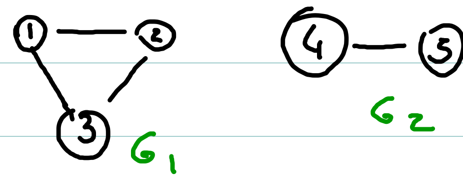
given any two vertices h, k in G_1 there is a path $h - i_1 \cdots i_m - k$

$w_h = w_{i_1} = \; w_{i_m} = w_k$ so $w_h = w_k$

So $w = \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix} = c\,\mathbb{1}$

Back to our exemple

G



Not connected

$$L_G = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}$$

$L_G u = 0 \iff u = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and

$L_{G_1} w_1 = 0$, $L_{G_2} w_2 = 0$

By previous th:

$w_1 = \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix} = c\,\mathbb{1}_3$ similarly $w_2 = \begin{bmatrix} d \\ d \end{bmatrix} = d\,\mathbb{1}_2$

so $u = \begin{bmatrix} c \\ c \\ c \\ d \\ d \end{bmatrix}$, therefore

so $u \in \text{span}\left(\begin{bmatrix} \mathbb{1}_3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbb{1}_2 \end{bmatrix} \right) = E_0$

so $\lambda = 0$ has multiplicity 2 in L_G

Th: G has k connected components $\iff \lambda_1 = 0$ has multeplicity k in L_G

Proof sketch: if G is connected $E_0 = \text{span}(\mathbb{1})$

$$\text{if } G = \underset{n_1 \text{ vertices}}{G_1} \quad \underset{n_2 \text{ vertices}}{G_2} \quad \cdots \quad \underset{n_k \text{ vertices}}{G_k}$$

$$L_G = \begin{bmatrix} L_{G_1} & & & 0 \\ & L_{G_2} & & \\ & & \ddots & \\ 0 & & & L_{G_k} \end{bmatrix} \text{ is a block matrix}$$

$$E_0 = \text{span}\left(\begin{bmatrix} \mathbb{1}_{n_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \mathbb{1}_{n_2} \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ \mathbb{1}_{n_k} \end{bmatrix} \right)$$