Lesson 14

Read chapter 6

Quadratic forms

Ex $\quad A = \begin{bmatrix} 7/6 & -1/3 & 1/6 \\ -1/3 & 5/3 & -1/3 \\ 1/6 & -1/3 & 7/6 \end{bmatrix}$

has eigenvalues $\lambda = 1, 1, 2$ with

$E_1 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$

$E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$.

Find an orthogonal diagonalization for A.

Describe $T \quad R^3 \to R^3 \quad T(v) = Av$

1) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is not orthonormal. Use Gram-Schmidt

$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

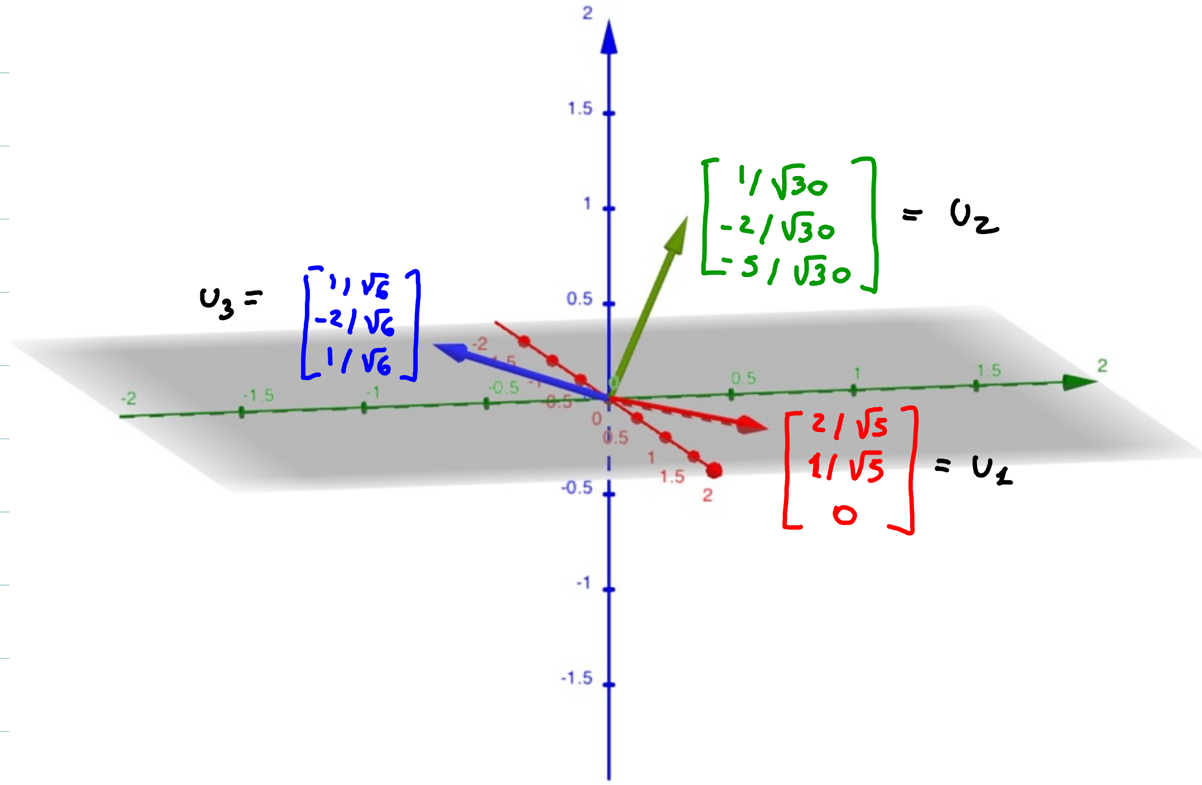
$e = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 2/5 \\ 1 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

$$B_1 = \begin{array}{cc} b_1 & b_2 \\ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$ not orthonormal use Gram Schmidt $\rightsquigarrow$ $\begin{array}{cc} u_1 & u_2 \\ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} & \begin{bmatrix} -1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \end{array}$

$$B_2 = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{b_3} \qquad \|v\| = \sqrt{1+4+1} = \sqrt{6} \rightsquigarrow \begin{array}{c} u_3 \\ \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \end{array}$$

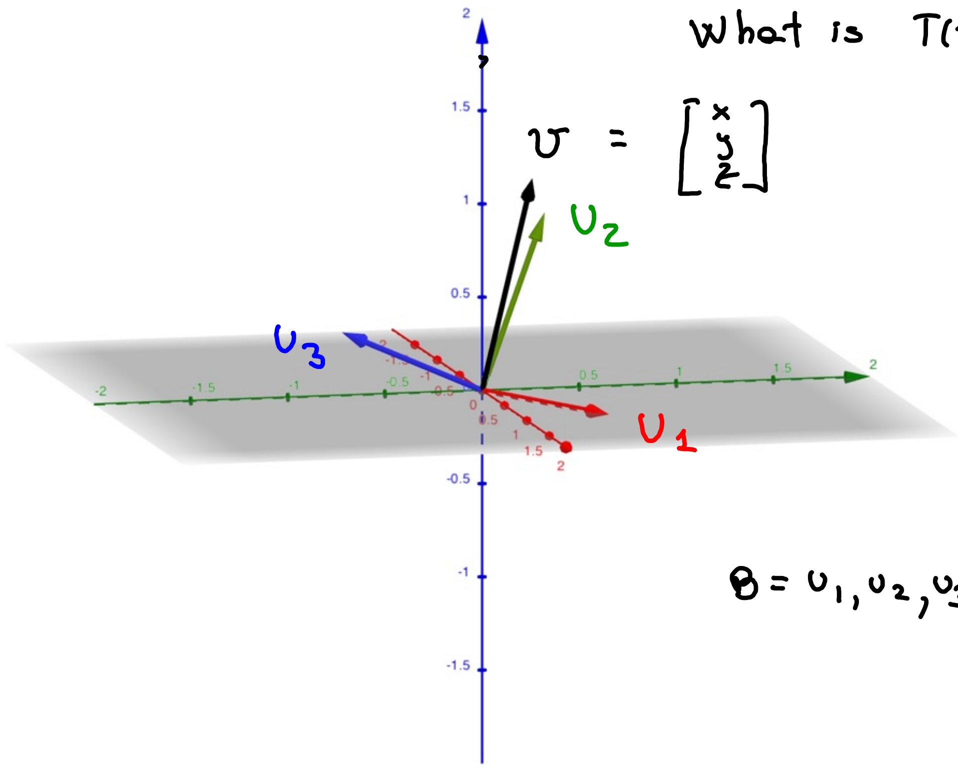
$$A = \overbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}}^{P} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{30} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}}^{P^{-1} = P^T}$$



$u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
$\begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} = u_2$
$\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = u_1$

$$A = \begin{bmatrix} 7/6 & -1/3 & 7/6 \\ -1/3 & 5/3 & -1/3 \\ 1/6 & -1/3 & 7/6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

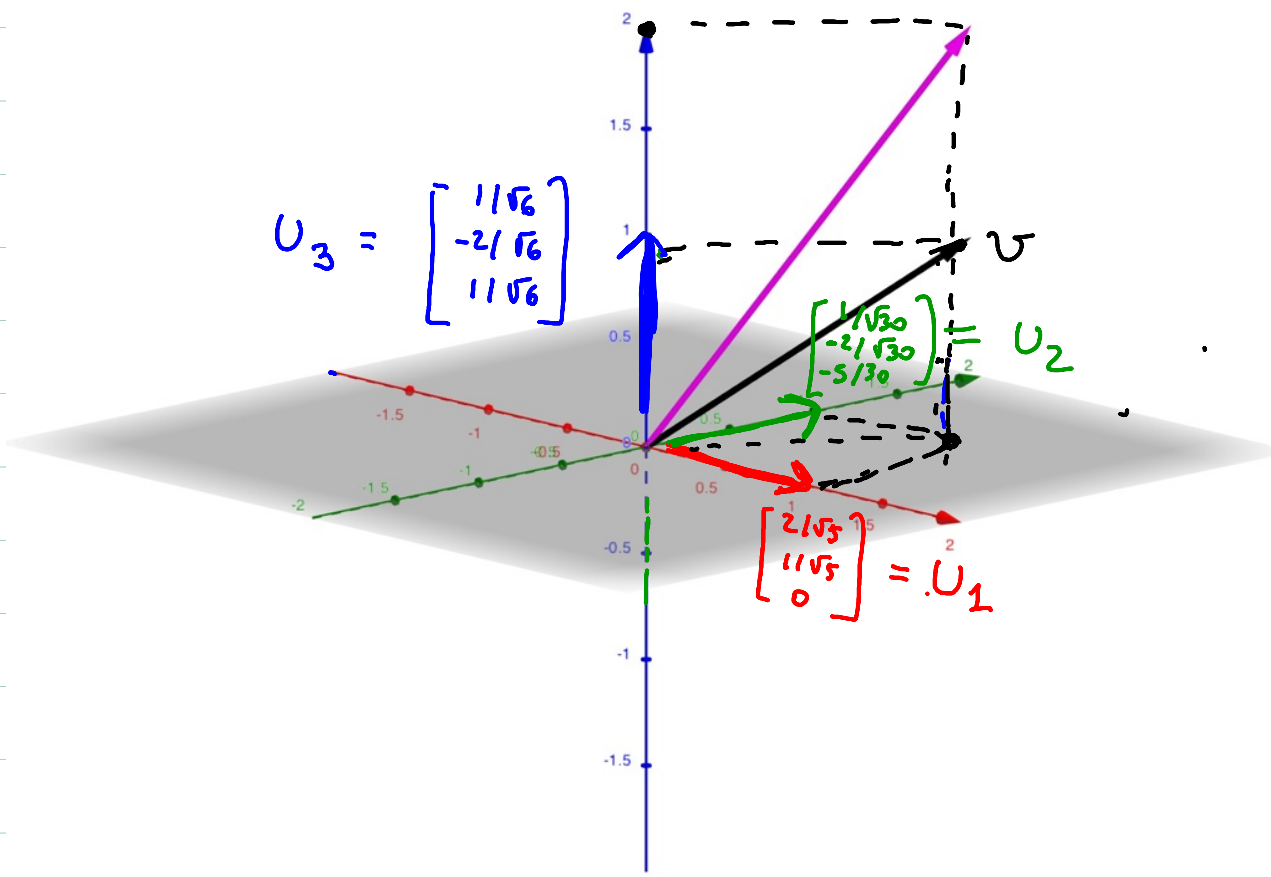
What is $T(v) = Av$?



$\mathcal{B} = u_1, u_2, u_3$

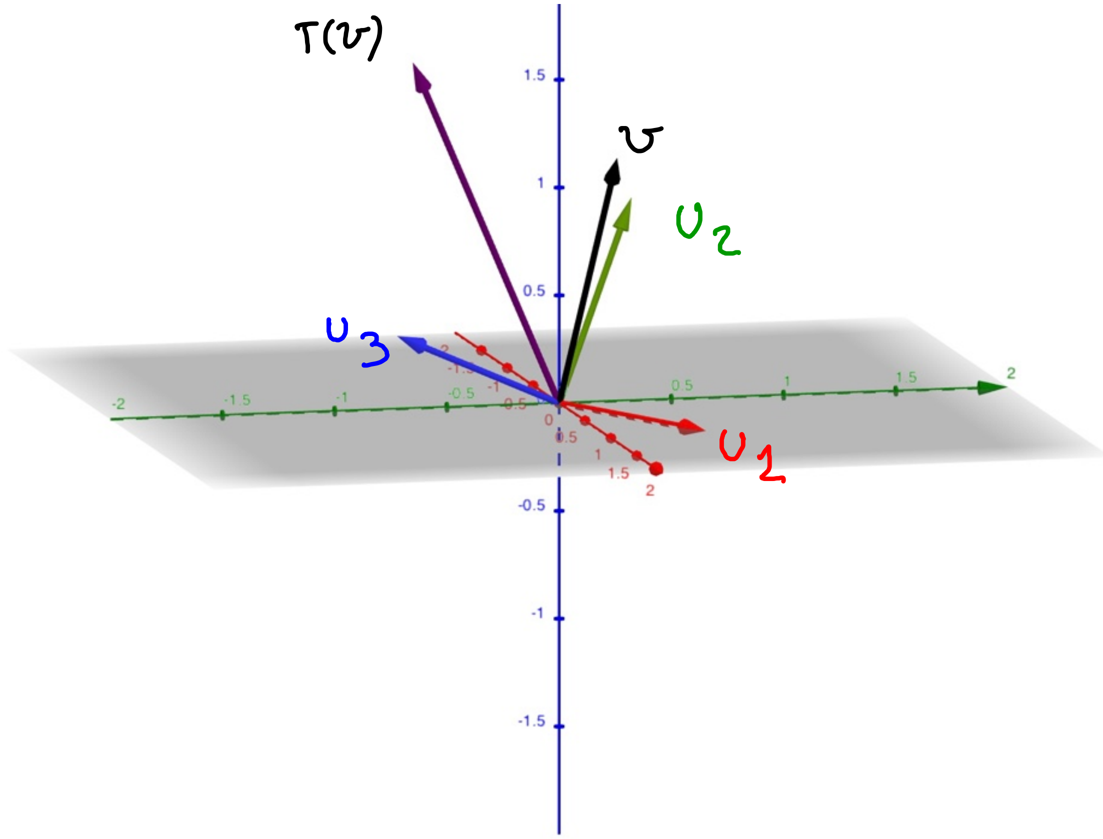
Recall $[T(v)]_{\mathcal{B}} = D[v]_{\mathcal{B}}$

If $[v]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ $\quad [T(v)]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \\ 2z' \end{bmatrix}$



$U_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
v
$\begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/30 \end{bmatrix} = U_2$
$\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = U_1$

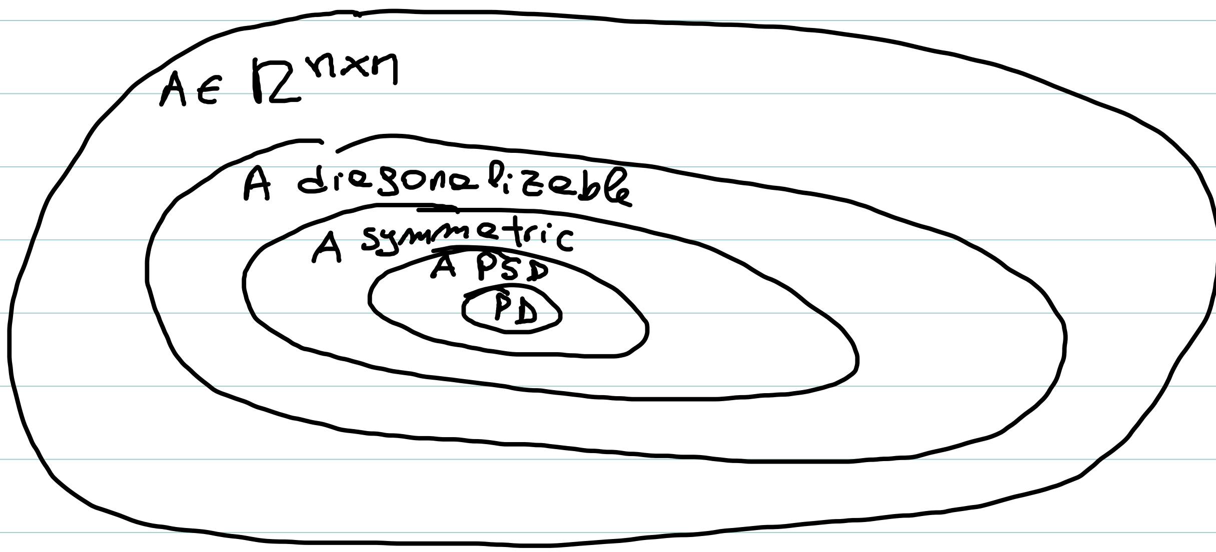
Back to x y z coordinate system



$$T(v) = u_1 + u_2 + 2u_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} + \begin{bmatrix} 2/\sqrt{6} \\ -4/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2\sqrt{6} + 1 + 2\sqrt{5}}{\sqrt{30}} \\ \dfrac{\sqrt{6} - 2 - 4\sqrt{5}}{\sqrt{30}} \\ \dfrac{-5 + 2\sqrt{5}}{\sqrt{30}} \end{bmatrix}$$

Fix n and consider $\mathbb{R}^{n\times n}$



$$P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^T$$

More terminology.

$p(x) = 3x^2 + 5x - 1$ polynomial

in one variable, it has degree 2

$p(x,y,z) = 3x^1y^1 - 5x^2 + 2y^2 + z^2 + 7z^1x^1$

polynomial in 3 variables, of degree 2

homogeneous

homogeneous = every monomial has degree 2

homogeneous polynomial = form

Quadratic forms

Given a symmetric matrix $A \in R^{n \times n}$

$[x_1 \cdots x_n] A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $(x^T A x)$ is a homogeneous polynomial of degree 2 in n variables i.e a

quadratic form.

Ex $n = 2$ $\underset{1\times 2}{[x \ \ y]} \underset{2\times 2}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = [x \ \ y] \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}$

$= ax^2 + 2bxy + cy^2$

$A = (a_{ij})$ symmetric means $a_{ij} = a_{ji}$

$$[x_1 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + \sum_{1 \le i < j \le n} 2a_{ij}x_ix_j$$

i.e monomial containing x_ix_j has coefficient $2a_{ij}$

Example

$$p(x,y,z) = 3xy - 5x^2 + 2y^2 + z^2 + 7zx$$
$$= 3x_1x_2 - 5x_1^2 + 2x_2^2 + x_3^2 + 7x_1x_3 =$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underset{3\times 3}{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

what is A?

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -5 & 3/2 & 7/2 \\ 3/2 & 2 & 0 \\ 7/2 & 0 & 1 \end{bmatrix} \end{array}$$

Quadratic forms in geometry

Consider $2x^2-4xy+5y^2=36$ what is this the equation of ?

$2x^2-4xy+5y^2$ is a quadratic form

Let's write it as $[x\ y]\ A\begin{bmatrix}x\\y\end{bmatrix}$

$$[x\ y]\underbrace{\begin{bmatrix}2 & -2\\-2 & 5\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix}=36$$

A is symmetric so A is orthogonally diagonalizable.

$\lambda = 1, 6 \qquad v_1 = \begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix} \qquad v_2 = \begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$

$B = v_1, v_2$ orthonormal basis for $\mathbb{R}^2$

$$A$$

$$(*)\ [x\ y]\ \overbrace{\underbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}}_{P^{-1} = P^T}} \begin{bmatrix} x \\ y \end{bmatrix} = 36$$

Recall change of basis stuff:

$P = \ddot{U}_B^E \qquad P^{-1} = P^T = U_E^B \qquad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P^{-1} v = [v]_B$$

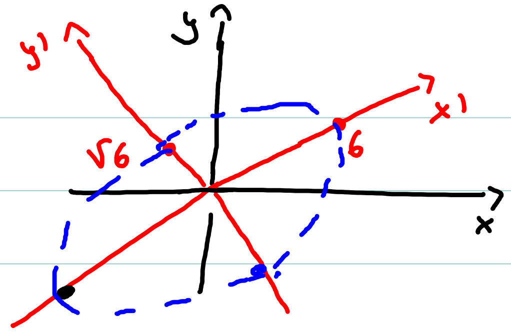
$$v^T P = v^T (P^T)^T = (P^T v)^T = (P^{-1} v)^T = [v]_B^T$$

$(*)$ becomes $[x\ y]_B \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_B = 36$

$$[x'\ y'] \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 36$$

$$x'^2 + 6y'^2 = 36$$

$$\frac{x'^2}{36} + \frac{y'^2}{6} = 1$$



$$\frac{1}{36} x'^2 + \frac{1}{6} y'^2 = 1$$

$$2x^2 - 4xy + 5y^2 = 36$$

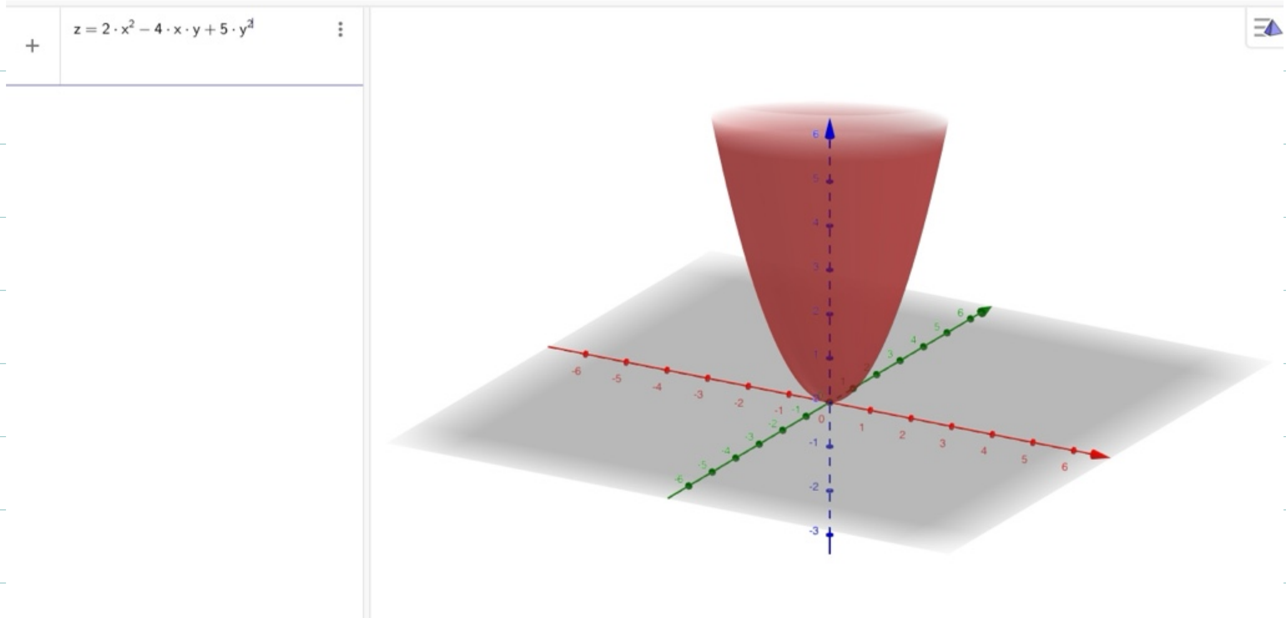
Note $\lambda = 1, 6 > 0$

Note for any $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have

$2x^2 - 4xy + 5y^2 > 0$ since given x, y.
there are some x' y' s.t

$$2x^2 - 4xy + 5y^2 = \frac{x'^2}{36} + \frac{y'^2}{6} \text{ and}$$

$$\frac{x'^2}{36} + \frac{y'^2}{6} > 0$$



z = 2 · x² − 4 · x · y + 5 · y²

$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ eigenvalues λ_1, λ_2

Are they positive?

$\det(A) = \lambda_1 \cdot \lambda_2 = 10 - 4 = 6 > 0$

both $\lambda_1, \lambda_2 > 0$ or both $\lambda_1, \lambda_2 < 0$

$\text{Trace}(A) = \lambda_1 + \lambda_2 = 2 + 5 > 0$

so $\lambda_1, \lambda_2 > 0$

Def : A principal minor of $A \in \mathbb{R}^{n\times n}$ is the determinant of the square submatrix of A obtained by removing any number k $0 \le k \le n-1$ of rows from A and the columns with the same indexes.

Ex
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

I could remove rows 1, 3 and columns 1, 3 so det $\begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix} = 42 - 40 = 2$

is a principal minor of A

How many principal minors in a 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$?

Remove 0 rows : det A

Remove 1 row :

$\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ removing row 1, col 1

$\det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$ removing row 2 col 2

$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ removing row 3 col 2

Remove 2 rows :

$\det(a_{11})$ removing row 2, 3 col 2,3

$\det(a_{22})$ removing row 1,3 col 1,3

$\det(a_{33})$ removing row 1,2 col 1,2

7 in total

Def: A $\boxed{\text{leading}}$ principal minor of $A \in \mathbb{R}^{n \times n}$ is the determinant of the square submatrix of A obtained by removing no rows or by removing rows $n, n-1, \dots k$ for some $n \geq k > 1$ and the columns with the same indexes.

Ex
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

I could remove rows 4, 3, 2 and columns 4, 3, 2 so $\det(1) = 1$ is a leading principal minor of A

How many leading principal minors in a 3x3 matrix?

remove no rows : $\det A$

remove row 3 : $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

remove rows 3 and 2 $\det [a_{11}]$

3 in total

Ex $\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

Leading principal minors : $10-4=6$, 2

Both positive

Principal but not leading minor : 5