Lesson 13

Read chapter 5

Symmetrix matrices: spectral theorem

Start chapter 6

orthonormal

Th: P is orthogonal $\Leftrightarrow$ $P^T \cdot P = I$ i.e $P^T = P^{-1}$

Note 1: if A, B are square matrices s.t. $BA = I$ then it must be true that $AB = I$

Note 2: An orthonormal set of vectors $c_1 \cdots c_m$ is always independent

Proof: suppose $\vec{c_1}, \vec{c_2}, \ldots \vec{c_m}$ are orthonormal and $h_1 \vec{c_1} + h_2 \vec{c_2} + \cdots + h_m \vec{c_m} = \vec{0}$ then

for $i = 1 \cdots m$ $\quad c_i^T \cdot (h_1 \vec{c_1} + \cdots + h_i \vec{c_i} + \cdots + h_m \vec{c_m}) = 0$

so $h_i = 0$ for $i = 1 \cdots m$.

Note 3: an orthonormal set $c_1 \cdots c_n$ consisting of n vectors in R^n is an <u>orthonormal</u> <u>basis</u> for R^n.

Note 4: if $c_1 \cdots c_n$ is an orthonormal basis for R^n then

$$b = b^T c_1 \cdot c_1 + b^T c_2 \, c_2 + \cdots + b^T c_n c_n$$ for all b in R^n

Proof: $b = x_1 c_1 + x_2 c_2 + \cdots x_i c_i + \cdots + x_n c_n$

$b_i^T \cdot b = x_i$

Fourier analysis

$V = f : R \to C$ continuous, with period 2π

$$f \cdot g = \int_{-\pi}^{\pi} f(x)\, \bar{g}(x)\, dx$$

$$e_n = \frac{1}{\sqrt{2\pi}} e^{inx} \qquad n = 0, \pm 1, \pm 2$$

orthonormal set idea

$$f = \sum_n f \cdot e_n \, e_n \quad ?$$

Th: $A \in \mathbb{R}^{n\times n}$ is orthogonally diagonalizable $\Leftrightarrow$ iff A is symmetric.
(spectral theorem)

Proof:

$\Rightarrow$: assume $A = PDP^{-1}$ with P orthogonal

then $A^T = (PDP^{-1})^T = (PDP^T)^T = PD^TP^T =$

$PDP^T = PDP^{-1} = A$

Proof: $\Leftarrow$ Suppose A is symmetric, and $\lambda_1, \lambda_2, \dots \lambda_k$ are the distinct eigenvalues of A

then $\lambda_1, \lambda_2, \dots \lambda_k \in \mathbb{R}$ (property of symmetric matrices).

Then $p(x) = (-1)^n(\lambda-\lambda_1)^{h_1}\cdots(\lambda-\lambda_k)^{h_k}$ and $h_1 + h_2 + \dots + h_k = n$ (True for any matrix).

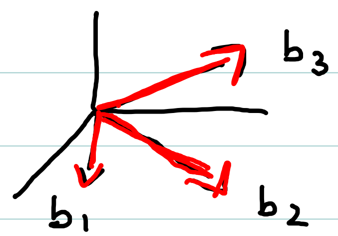
Also $GM(\lambda_i) = AM(\lambda_i) = h_i$ (true for symmetric matrices) Let B_i be an orthonormal basis for E_{λ_i} The set $B_1 \cup B_2 \cup \dots \cup B_k = B$ is linearly independent (because it is orthonormal) and has $h_1 + h_2 + \dots + h_k = n$ vectors so it is a basis for $\mathbb{R}^n$. Take $P = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$ where $B = b_1 \dots b_n$ then

P is ~~orthogonal~~ orthonormal and $A = PDP^{-1}$

Th: Given $V \leq \mathbb{R}^n$ ($V = \mathbb{R}^n$ ok)
and a basis $B: b_1, \dots b_k$ for V we
can always construct from B a new
orthonormal basis $u_1, \dots u_k$ for V

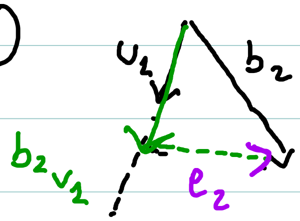
(Gram-Schmidt algorithm)

Proof idea: suppose b_1, b_2, b_3 is a basis for $\mathbb{R}^3$



1) $u_1 = \dfrac{b_1}{\|b_1\|}$ $\quad V_1 = \text{span}(u_1) = \text{span}(b_1)$

2)



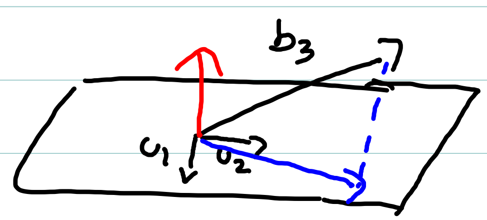
Project b_2 on $\text{span}(u_1) = V_1$

$b_{2V_1}: \; u_1^T b_2 \cdot u_1$

take $e_2 = b_2 - u_1^T b_2 \, u_1$

2) $u_2 = \dfrac{e_2}{\|e_2\|}$, $\quad V_2 = \text{span}(u_1, u_2) = \text{span}(b_1, b_2) =$

3)



Project b_3 on $\text{span}(u_1 \; u_2) = V_2$

$b_{3V_2} = u_1^T b_3 \cdot u_1 + u_2^T b_3 \cdot u_2$

take

$e_3 = b_3 - u_1^T b_3 \cdot u_1 - u_2^T b_3 \cdot u_2$

$u_3 = \dfrac{e_3}{\|e_3\|}$

$\text{span}(u_1, u_2, u_3) = \text{span}(b_1, b_2, b_3)$

Ex : from hw 1

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \lambda = 0, 1, 3$$

$$E_0 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \qquad E_1 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \qquad E_3 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is orthogonal not orthonormal

$\|v_1\| = \sqrt{3}$ $\qquad \|v_2\| = \sqrt{2}$ $\qquad \|v_3\| = \sqrt{6}$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}_{P^{-1} = P^T}$$

Not done in class

Def: if $z = a + ib \in \mathbb{C}$ the **the conjugate** of z, $\bar{z}$ is $\bar{z} = a - ib$.

Def: If $A = (a_{ij})$ is a matrix then $\bar{A} = (\bar{a}_{ij})$

If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{C}^n$ then $\bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$

Th: if $z_1, z_2 \in \mathbb{C}$ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

if $v \in \mathbb{C}^n$ $A \in \mathbb{C}^{m \times n}$ $\overline{Av} = \bar{A} \cdot \bar{v}$

Note: λ is real $\Leftrightarrow \bar{\lambda} = \lambda$ since

$\lambda = a + ib$ $\bar{\lambda} = a - ib$

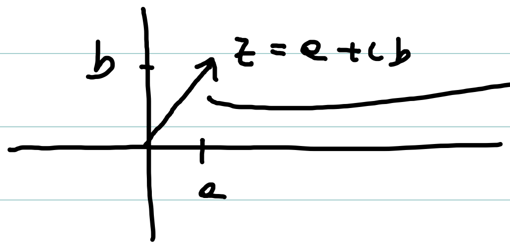
$\lambda = \bar{\lambda} \Leftrightarrow b = 0$ ie $\lambda \in \mathbb{R}$

Not done in class

Def: If $z \in \mathbb{C}$, $z = a + ib$, $\|z\| = \sqrt{z \cdot \bar{z}}$

$$= \sqrt{(a+ib)(a-ib)} = \sqrt{a^2 - (ib)^2} =$$

$$= \sqrt{a^2+b^2} \in \mathbb{R}$$



$\|z\| = \sqrt{a^2+b^2}$

Note: $\|z\| = 0 \iff z = 0$

Def: if $v \in \mathbb{C}^n$, $\|v\| = \sqrt{\bar{v}^T \cdot v}$

(so for $v, w \in \mathbb{C}^n$ $v \cdot w = \bar{v}^T w$)

Not done in class

Example:

$$A=\begin{pmatrix}1 & i\\ 1+i & 0\end{pmatrix} \qquad v=\begin{pmatrix}1+i\\ 2-i\end{pmatrix} \qquad \bar{A}=\begin{pmatrix}1 & -i\\ 1-i & 0\end{pmatrix} \qquad \bar{v}=\begin{bmatrix}1-i\\ 2+i\end{bmatrix}$$

$$Av=\begin{pmatrix}1 & i\\ 1+i & 0\end{pmatrix}\begin{pmatrix}1+i\\ 2-i\end{pmatrix}=\begin{pmatrix}1+i+2i-i^2\\ 1+2i+i^2\end{pmatrix}=\begin{pmatrix}2+3i\\ 2i\end{pmatrix}$$

$$\bar{A}\bar{v}=\begin{pmatrix}1 & -i\\ 1-i & 0\end{pmatrix}\begin{pmatrix}1-i\\ 2+i\end{pmatrix}=\begin{pmatrix}1-i-2i-i^2\\ 1-2i+i^2\end{pmatrix}=\begin{pmatrix}2-3i\\ -2i\end{pmatrix}$$

$$\overline{Av}=\bar{A}\bar{v}$$

Not done in class

Th: The eigenvalues and eigenvectors of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ are real.

Property 1

Proof idea: Assume λ is an eigenvalue for A and show $\lambda = \bar{\lambda}$. This tells us $\lambda \in \mathbb{R}$

Use $A = \bar{A}$ (because $A \in \mathbb{R}^{n\times n}$)

and $A = A^T$

Proof: suppose $Av = \lambda v$, then $\overline{Av} = \overline{\lambda v}$

therefore $\bar{A}\bar{v} = \bar{\lambda}\bar{v}$ so $A\bar{v} = \bar{\lambda}\bar{v}$

(because A is real) so $v^T A \bar{v} = v^T \bar{\lambda} \bar{v}$

and $v^T A^T \bar{v} = \bar{\lambda} v^T \bar{v}$ (because $A = A^T$)

so $(Av)^T \bar{v} = \bar{\lambda} v^T \bar{v}$ and

$(\lambda v)^T \bar{v} = \bar{\lambda} v^T \bar{v}$ so

$\lambda v^T \bar{v} = \bar{\lambda} v^T \bar{v}$ since $v \neq 0$

$v^T \bar{v} \neq 0$ because if $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,

$$v^T \bar{v} = (x_1 \cdots x_n)\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} x_1\bar{x}_1 \\ \\ x_1\bar{x}_n \end{pmatrix} = \begin{pmatrix} \|x_1\|^2 \\ \vdots \\ \|x_n\|^2 \end{pmatrix}$$

so $\lambda = \bar{\lambda}$

Th If $A \in \mathbb{R}^{n \times n}$ is symmetric and v, w are eigenvectors for distinct eigenvalues λ, μ then $v \perp w$

Property 2

Idea: use $A = A^T$ to show $v^T \cdot w = 0$

Proof: we know that $Av = \lambda v$, $Aw = \mu w$

then $v^T A w = v^T \mu w = \mu v^T w$ but also

$v^T A w = v^T A^T w = (Av)^T w = \lambda v^T w$

$\lambda v^T w = \mu v^T w$ and since $\lambda \neq \mu$

then $v^T w = 0$

Th: if $A \in \mathbb{R}^{n\times n}$ is symmetric and λ is an eigenvalue of A, then $AM(\lambda) = GM(\lambda)$

Property 3

Proof (not covered in class, you do not have to know this)

Let $b_1 \cdots b_r$ be an orthonormal basis for E_λ^A

so $GM(\lambda) = r$. Our goal is to show $GM(\lambda) = r$.

complete $b_1 \cdots b_r$ to an orthonormal basis

$b_1 \cdots b_r \; b_{r+1} \cdots b_n$ for $\mathbb{R}^n$

Then $A b_j = \lambda b_j$ if $1 \le j \le r$

$A b_j = \sum_{l=1}^{n} c_{lj} b_l$ if $r+1 \le j \le n$ for some scalars c_{lj}

Therefore

$$A \underbrace{\begin{bmatrix} b_1 \cdots b_r & b_{r+1} \cdots b_n \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} b_1 \cdots b_r & b_{r+1} \cdots b_n \end{bmatrix}}_{S} \begin{bmatrix} \lambda & & & c_{1\,r+1} & & c_{1\,n} \\ & \ddots & & \vdots & & \\ & & \lambda & c_{r\,r+1} & & c_{r\,n} \\ & & & c_{n\,r+1} & & c_{n\,n} \end{bmatrix}$$

$$\text{So} \quad S^{-1} A S = \begin{bmatrix} \lambda & & & c_{1\,r+1} & & c_{1\,n} \\ & \ddots & & \vdots & & \\ & & \lambda & c_{r\,r+1} & & c_{r\,n} \\ & & & c_{n\,r+1} & & c_{n\,n} \end{bmatrix}$$

since $b_1\, b_2 \cdots b_n$ is orthonormal $S^{-1} = S^T$

$$\text{So} \quad S^T A S = \begin{bmatrix} \lambda & & & c_{1\,r+1} & & c_{1\,n} \\ & \ddots & & \vdots & & \\ & & \lambda & c_{r\,r+1} & & c_{r\,n} \\ & & & c_{n\,r+1} & & c_{n\,n} \end{bmatrix}$$

$(S^T A S)^T = S^T A^T S = S^T A S$ since $A = A^T$

Therefore $\begin{bmatrix} \lambda & & & c_{1\,r+1} & & c_{1\,n} \\ & \ddots & & \vdots & & \\ & & \lambda & c_{r\,r+1} & & c_{r\,n} \\ & & & c_{n\,r+1} & & c_{n\,n} \end{bmatrix}$

is symmetric so the entries c_{ij} for $1 \le i \le r$ must be 0

so $S^{-1} A S = \left[\begin{array}{c|c} \begin{matrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{matrix} & 0 \\ \hline 0 & C \end{array}\right] = B$

$\det(A - xI) = \det(S^{-1}(A - xI)S)$ (similar matrices have the same characteristic polynomial)

$= \det(S^{-1}AS - xI) = \det \left[\begin{array}{c|c} \begin{matrix} \lambda - x & & \\ & \ddots & \\ & & \lambda - x \end{matrix} & 0 \\ \hline 0 & C - xI \end{array}\right]$

$= (\lambda - x)^r \det(C - xI)$

To show $r = GM_A(\lambda) = AM_A(\lambda)$ we need to show λ is not an eigenvalue for C: suppose by contradiction $Cu = \lambda u$ for some $u \ne \vec{0}$ then

$\underbrace{\left[\begin{array}{c|c} \begin{matrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{matrix} & 0 \\ \hline 0 & C \end{array}\right]}_{B} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix}$ therefore

Since $A = SBS^{-1}$

$$AS\begin{bmatrix}0\\ \vdots\\ u\end{bmatrix} = SBS^{-1}S\begin{bmatrix}0\\ \vdots\\ 0\\ u\end{bmatrix} = SB\begin{bmatrix}0\\ \vdots\\ u\end{bmatrix} = \lambda S\begin{bmatrix}0\\ \vdots\\ 0\\ u\end{bmatrix}$$

$$S\begin{bmatrix}0\\ \vdots\\ 0\\ u\end{bmatrix} = \begin{bmatrix}\overbrace{b_1 \cdots b_r}^{\text{basis for } E_\lambda^A} & b_{r+1} \cdots & b_n\end{bmatrix}\begin{bmatrix}\left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\} r\\ u\end{bmatrix} =$$

$$\begin{bmatrix}b_1 \cdots b_r & b_{r+1} \cdots & b_n\end{bmatrix}\begin{bmatrix}\left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\} r\\ \left.\begin{matrix}u_1\\ \vdots\\ u_{n-r}\end{matrix}\right\} u\end{bmatrix} =$$

$$u_1 \vec{b}_{r+1} + \cdots u_{n-r} \vec{b}_n$$

Therefore $S\begin{bmatrix}0\\ \vdots\\ u\end{bmatrix}$ is an eigenvector of A

for eigenvalue λ that is not in span

of $b_1, \cdots b_r$, but $b_1 \cdots b_r$ is a

basis for E_λ so this is impossible.

Therefore we can conclude λ is not an

eigenvalue for C.

end of proof.

Ex $A = \begin{bmatrix} 7/6 & -1/3 & 1/6 \\ -1/3 & 5/3 & -1/3 \\ 1/6 & -1/3 & 7/6 \end{bmatrix}$

has eigenvalues $\lambda = 1, 1, 2$ with

$E_1 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$

$E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$

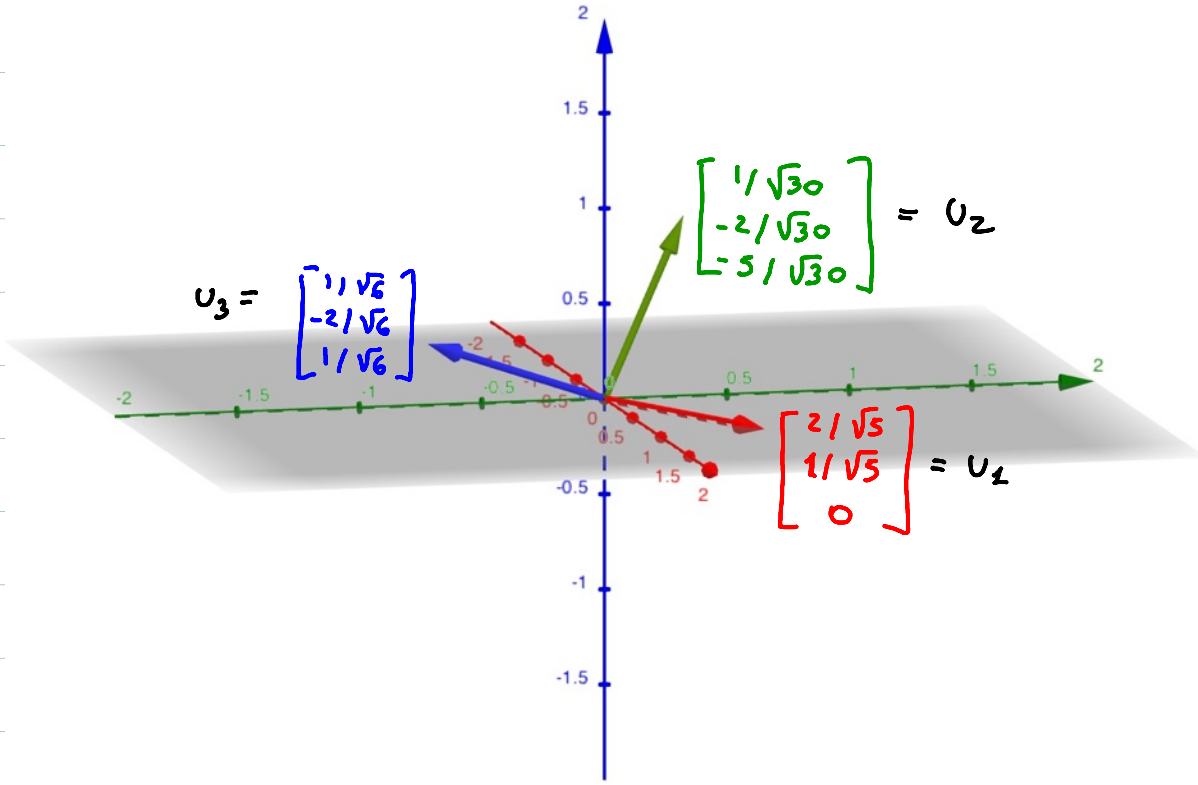
Find an orthogonal diagonalization for A.

Describe $T \quad R^3 \to R^3 \quad T(v) = Av$

$$B_1 = \overset{b_1}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}} \overset{b_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \quad \text{not orthonormal} \quad \text{use Gram Schmidt} \rightsquigarrow \overset{u_1}{\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}} \overset{u_2}{\begin{bmatrix} -1/\sqrt{30} \\ 2/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix}}$$

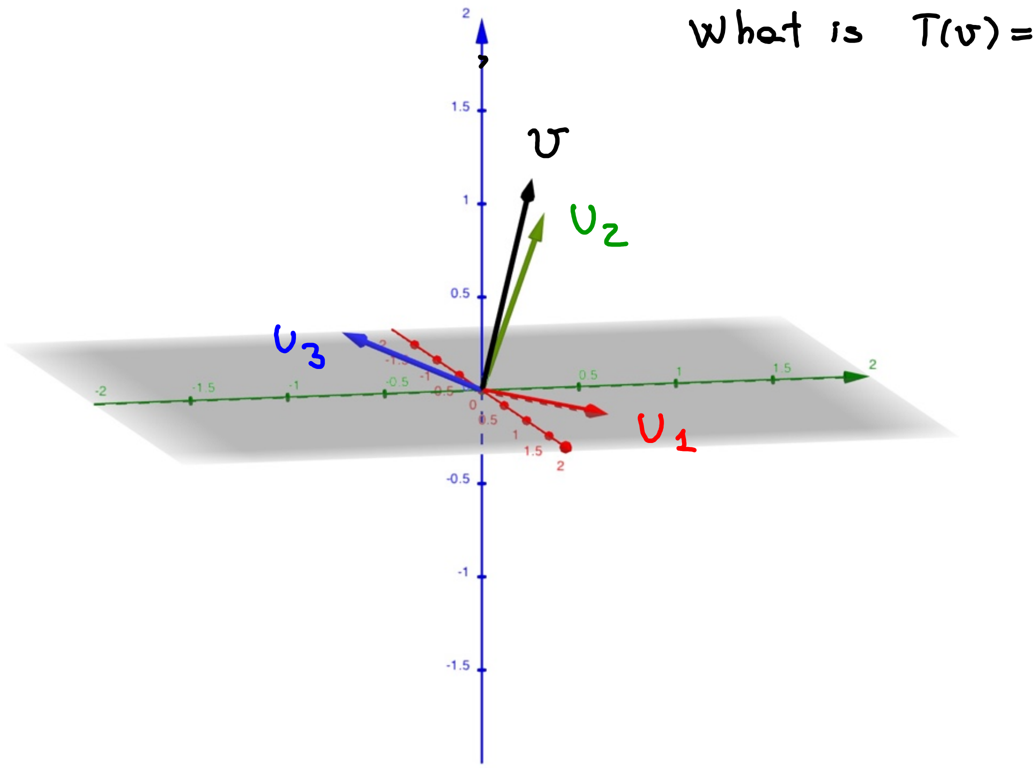
$$B_2 = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{b_3} \quad \|v\| = \sqrt{1+4+1} = \sqrt{6} \rightsquigarrow \overset{u_3}{\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}}$$

$$A = \overbrace{\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}}^{P} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{30} & 2/\sqrt{30} & -2/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \end{bmatrix}}^{P^{-1} = P^T}$$


$u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
$\begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} = u_2$
$\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} = u_1$

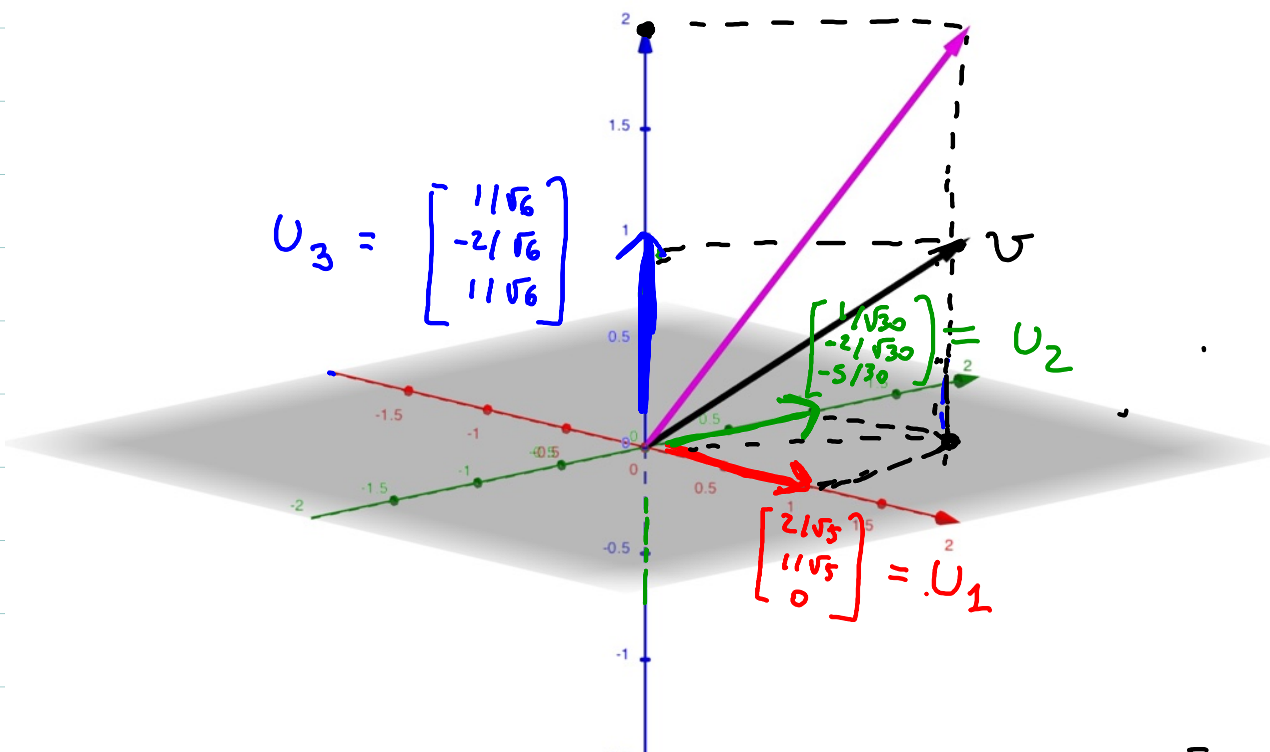
$$A = \begin{bmatrix} 7/6 & -1/3 & 7/6 \\ -1/3 & 5/3 & -1/3 \\ 1/6 & -1/3 & 7/6 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

What is $T(v) = Av$?



Recall $[T(v)]_B = D[v]_B$

$B = u_1, u_2, u_3$



$B \quad u_1 \quad u_2 \quad u_3$

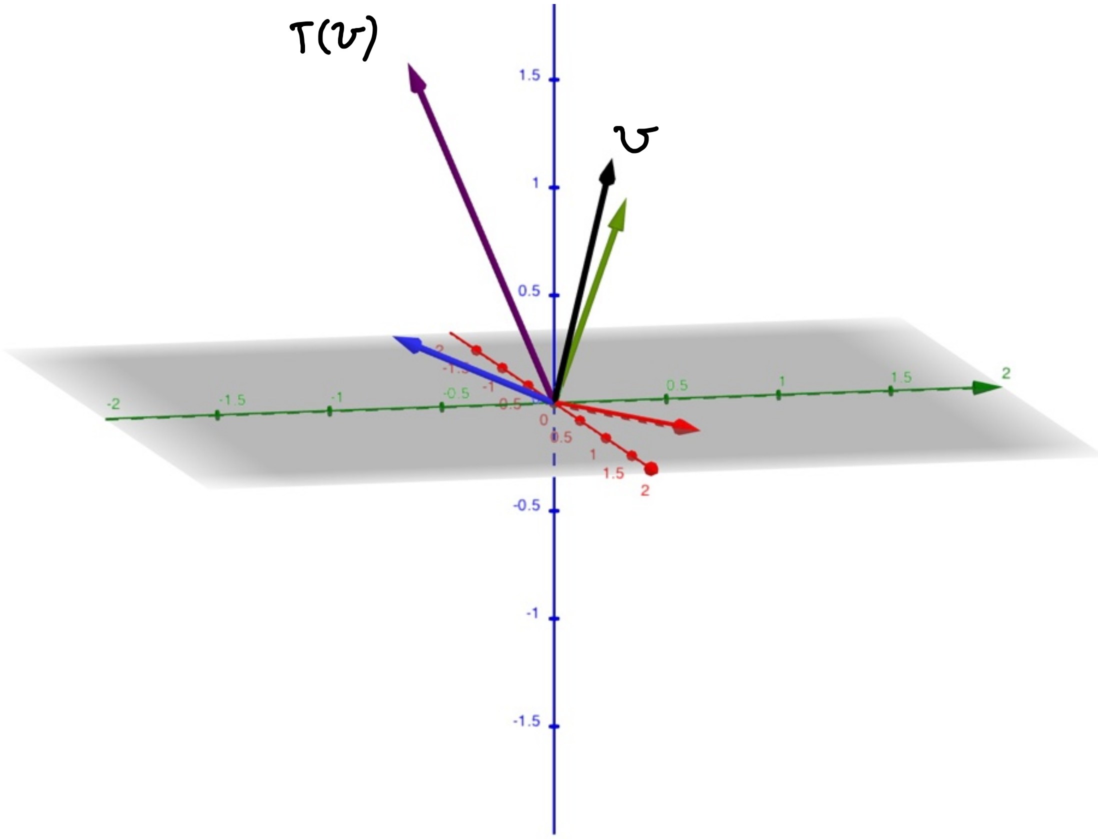
$[v]_B \quad [Tv]_B$

$$v = u_1 + u_2 + u_3$$

$$T(v) = u_1 + u_2 + 2u_3$$

$$D \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Back to xyz coordinate system



$$T(v) = v_1 + v_2 + 2v_3 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 1/\sqrt{30} \\ -2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix} + \begin{bmatrix} 2/\sqrt{6} \\ -4/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2\sqrt{6}+1+2\sqrt{5}}{\sqrt{30}} \\ \dfrac{\sqrt{6}-2-4\sqrt{5}}{\sqrt{30}} \\ \dfrac{-5+2\sqrt{5}}{\sqrt{30}} \end{bmatrix}$$

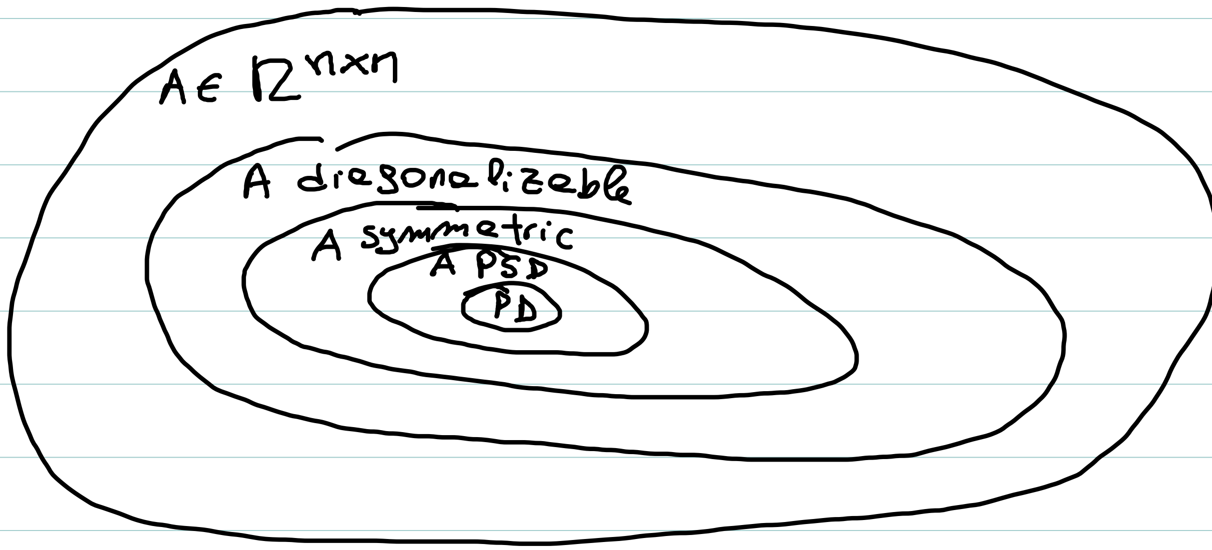
Ch 6

Recall every symmetric matrix A is orthogonally diagonalizable.

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^T, \quad P \text{ orthogonal}.$$

Sometimes it is useful to know whether all λ_i are positive or negative.

Fix n and consider $R^{n \times n}$



More terminology.

$p(x) = 3x^2 + 5x - 1$ polynomial

in one variable, it has degree 2

$p(x,y,z) = 3x^1y^1 - 5x^2 + 2y^2 + z^2 + 7z^1x^1$

polynomial in 3 variables, of degree 2

homogeneous

homogeneous = every monomial has degree 2

homogeneous polynomial = form

Quadratic forms

Given a symmetric matrix $A \in R^{n \times n}$

$[x_1 \cdots x_n] A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $(x^T A x)$ is a homogeneous polynomial of degree 2 in n variables i.e a

quadratic form.

Ex $n = 2$ $\underset{1 \times 2}{[x \ \ y]} \underset{2 \times 2}{\begin{bmatrix} a & b \\ b & c \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = [x \ \ y] \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}$

$= ax^2 + 2bxy + cy^2$

$A = (a_{ij})$ symmetric means $a_{ij} = a_{ji}$

$$[x_1 \cdots x_n] \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \cdots & & & & \\ \cdots & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots$$

$$= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + \sum_{1 \le i < j \le n} 2a_{ij}x_ix_j$$

i.e monomial containing x_ix_j has coefficient $2a_{ij}$