Lesson 10

Read chapter 6,

Projections

Def: Given $b \in \mathbb{R}^n$ and $V \leq \mathbb{R}^n$

we know that we can write

$$b = b_V + e \quad \text{with } b_V \in V, \ e \in V^\perp$$

in a unique way.

b_V is called the orthogonal projection of b on V.

$$P_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$P_V(b) = b_V$$

is a linear transformation, therefore

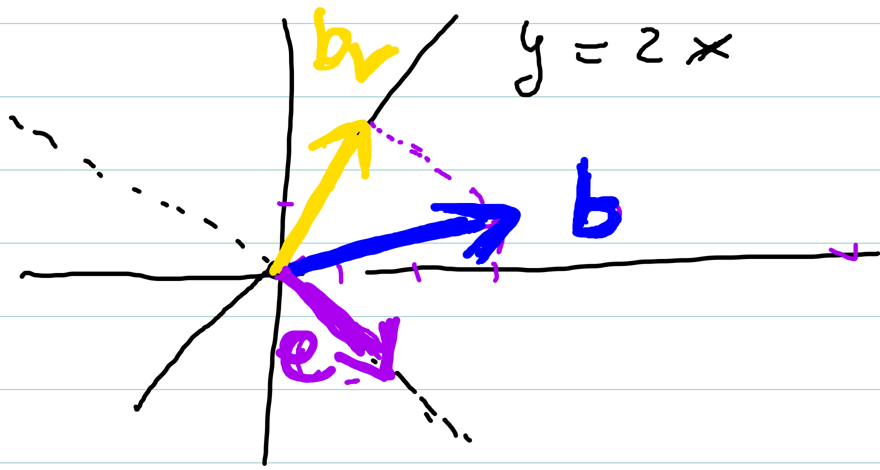
$$P_V(b) = P\,b$$

for some matrix P

Does P have special properties?
(recall hw 2 # 2)

How do we compute P ?

Example $V = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \leq \mathbb{R}^2$ $\quad V^\perp = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$



Note : if $b \in V$ $P_V(b) = b$

if $b \in V^\perp$ $P_V(b) = \vec{0}$

Properties of P :

1) $P^2 = P$

$P_V(P_V(b)) = P_V(b_V) = b_V$

$P^2 b = b_V = P b$

for all b in R^n so $P^2 = P$

2) $V = \mathrm{col}(P) = \mathrm{Range}(P_V)$

3) $\mathrm{Null}(P) = V^\perp$

4) $P^T = P$

Proof :

Given x, y in R^n

$x = v_1 + w_1$ $v_1 \in V$ $w_1 \in V^\perp$

$y = v_2 + w_2$ $v_2 \in V$ $w_2 \in V^\perp$

$x^T P y = (v_1^T + w_1^T) v_2 = v_1^T v_2$

$x^T P^T y = (P x)^T y = v_1^T (v_2 + w_2) = v_1^T v_2$

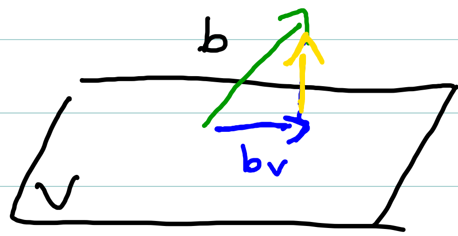
Therefore $P = P^T$.

Projections

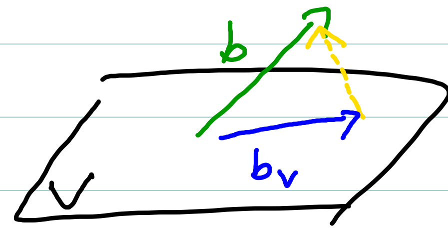
$P_V \quad R^n \longrightarrow R^n$

$P^2 \stackrel{?}{=} P$

$col(P) = V$



orthogonal



oblique

$NULL(P) = V^\perp$

$P = P^T$

Th : Suppose $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = n$
then $A^T \cdot A$ is invertible.

A, A^T may not even be square so we cannot consider A^{-1}, $(A^T)^{-1}$ but $A^T A$ is invertible why?

Proof : . $\therefore$ $A^T A \in \mathbb{R}^{n \times n}$
we want to show $\text{NULL}(A^T A) = \{\vec{0}\}$
then $\text{rank}(A^T A) = n - \text{nullity}(A^T A) = n$
so $A^T A$ is invertible.
Suppose $A^T A y = 0$ then $y^T A^T A y = 0$
so $\|Ay\|^2 = 0$ so $Ay = \vec{0}$ so
$y \in \text{Null}(A)$. Since $\text{rank}(A) = n$
$\text{nullity}(A) = n - n = 0$ so $y = \vec{0}$

Let $V \leq \mathbb{R}^n$, $B = b_1 \cdots b_k$ a basis for V

Let $A = \begin{bmatrix} b_1 \cdots b_k \end{bmatrix}$ $\quad A \in \mathbb{R}^{n \times k}$

A has rank k

Our goal : find P s.t $Pb = b_V$ for all b

we know $b_V = x_1 b_1 + x_2 b_2 + \cdots + x_k b_k$

for some scalers $x_1\ x_2 \cdots x_k$, let $\hat{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$

then $A\hat{x} = b_V$. How do we find $\hat{x}$?

If B orthonormal (i.e $b_i^T b_j = 0$

for $i \neq j$ $\quad b_i^T b_i = 1$)

then $b_i^T b_V = b_i^T (x_1 b_1 + \cdots x_i b_i + \cdots + x_k b_k) = x_i$

So $A^T b_V = \begin{bmatrix} b_1^T \\ \vdots \\ b_k^T \end{bmatrix} b_V = \hat{x}$

If B is not orthonormal ?

$$A^T b_V = A^T A \hat{x}$$

Note that $\quad b = b_V + e \quad$ therefore

$$b_i^T b = b_i^T (b_V + e) = b_i^T b_V$$

So $\quad A^T b = A^T b_V = A^T A \hat{x}$

by previous th $A^T A$ is invertible

$$(A^T A)^{-1} A^T b = \hat{x}$$

$$\underbrace{A(A^T A)^{-1} A^T}_{P} b = A\hat{x} = b_V$$

Let's check $P^2 = P$

$$A \underbrace{(A^T A)^{-1} A^T A}_{I} (A^T A)^{-1} A^T = A (A^T A)^{-1} A^T$$

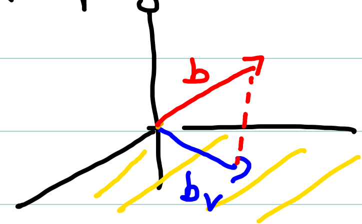
$P^T = P$

$$(A (A^T A)^{-1} A^T)^T = A \left((A^T A)^{-1}\right)^T A^T$$
$$= A \left((A^T A)^T\right)^{-1} A^T = A (A^T A)^{-1} A^T$$

Example

Given $V = xy$ plane in $\mathbb{R}^3$, find P

(matrix of orthogonal projection on V)



1) Find basis for V : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Notice that this basis is orthonormal: all vectors in it have length 1 and are pairwise orthogonal

2) Write $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3) $P = A(A^T A)^{-1} A^T$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{I_2} \Bigg)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Easy because basis was orthonormal

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P \; ; \quad P \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

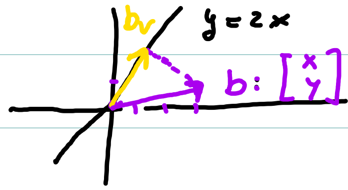
Check that $P^2 = P$ $\quad P = P^T$

Note that $Pb = A A^T b = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} u_1^T b \\ u_2^T b \end{bmatrix} =$

$$= u_1^T b \cdot u_1 + u_2^T b \; u_2$$

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ orthonormal basis for V

Orthogonal projection on line $y = 2x$:



1) Find a basis for V

$$B = \begin{bmatrix}1\\2\end{bmatrix}$$

2) $A = \begin{bmatrix}1\\2\end{bmatrix}$

3) $$P = \begin{bmatrix}1\\2\end{bmatrix}\Bigg(\underbrace{\underset{1\times 2}{\begin{bmatrix}1&2\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}1\\2\end{bmatrix}}}_{1\times 1}\Bigg)^{-1}\begin{bmatrix}1&2\end{bmatrix}$$

$$P = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\underset{1\times 2}{\begin{bmatrix}1&2\end{bmatrix}} = \frac{1}{5}\begin{bmatrix}1&2\\2&4\end{bmatrix};$$ so for example

The orthogonal projection of $\begin{bmatrix}4\\1\end{bmatrix}$ is :

$$P\begin{bmatrix}4\\1\end{bmatrix} = \frac{1}{5}\overset{2\times 1}{\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}} = \begin{bmatrix}6/5\\12/5\end{bmatrix};$$

we could also write

$$P\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix} \quad ; \quad 5 = \left\|\begin{bmatrix}1\\2\end{bmatrix}\right\|^2$$

$$\frac{1}{\|b_1\|^2}\, b_1^T\begin{bmatrix}x\\y\end{bmatrix}\cdot b_1$$

$$\left(\frac{b_1}{\|b_1\|}\right)^T\begin{bmatrix}x\\y\end{bmatrix}\cdot\frac{b_1}{\|b_1\|}$$

Note: $\mathrm{Null}(P) = \mathrm{span}\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)^{\perp}$

In general the orthogonal projection in $\mathbb{R}^n$ of b on line through origin parallel to b_1 $(V = \mathrm{span}(b_1))$ is:

$$Pb = \frac{1}{\|b_1\|^2} b_1^T b \cdot b_1$$

if $\|b_1\| = 1$ $\quad Pb = b_1^T b \cdot b_1$

Example:

Given $P = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$.

Is it the matrix of a projection? in hw "projector"? $P = P^2$ yes

Is it orthogonal? $P = P^T$? NO

Where are we projecting? $V = \text{Col}(P) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

How are we projecting? In the direction . . of $\text{Null}(P) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

Note $\text{NULL}(P) \not\perp \text{Col}(P)$, but $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basis for $\mathbb{R}^2$

Therefore any b in $\mathbb{R}^2$ can be written as:

$$b = c_1 \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\in \text{Col}(P)} + c_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\in \text{NULL}(P)}$$