Lesson 1

Read ch 1

Eigenvalues, eigenvectors,
characteristic polynomial
Eigenspaces, multiplicities
Change of bases.

Def: given $A \in \mathbb{R}^{n \times n}$ or $T: \mathbb{R}^n \to \mathbb{R}^n$

if $v \neq 0$ and

$Av = \lambda v$ $\qquad$ $T(v) = \lambda v$

then λ is an eigenvalue

v is an eigenvector

for A/T

- How to calculate
- Characteristic polynomial
- Geometric interpretation
- Applications

$Av = \lambda v \iff Av - \lambda v = \vec{0} \iff (A - \lambda I)v = \vec{0}$

$\iff v \in Null(A - \lambda I)$

$Null(A - \lambda I) \neq \{\vec{0}\} \iff \det(A - \lambda I) = 0$

$\det(A - \lambda I) = p(\lambda)$ charecteristic polynomial of A

$$\boxed{\text{the eigenvalues of } A \text{ are the roots of } p(\lambda) \text{ ie the solutions of } p(\lambda) = 0}$$

If $A \in \mathbb{R}^{n \times n}$ $\quad p(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

in hw2: if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A

$a_0 = \lambda_1 \cdot \ldots \cdot \lambda_n = \det(A)$

$a_{n-1} = (-1)^{n-1} \underbrace{(a_{11} + a_{22} + \cdots + a_{nn})}_{\text{Trace of } A} = (-1)^{n-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

Given $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) and we are in $\mathbb{C}$, $p(\lambda) = 0$ has n solutions

$\lambda_1 \cdots\cdots \lambda_n$, not necessarily distinct

$$p(\lambda) = (-1)^n (\lambda - \lambda_{i_1})^{k_1} (\lambda - \lambda_{i_2})^{k_2} \cdots (\lambda - \lambda_{i_r})^{k_r}$$

if $\lambda_{i_1}, \lambda_{i_2}, \dots \lambda_{i_r}$ are the r distinct solutions to $p(\lambda) = 0$

λ_{i_j} has algebraic multiplicity equal to k_j

$k_1 + k_2 + \cdots + k_r = n$

Ex $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

$$p(\lambda) = (\lambda - i)^2 (\lambda + i)^2$$

What happens in $\mathbb{R}$?

$A \in \mathbb{R}^{n\times n}$, $p(\lambda) = (-1)^n (\lambda-\lambda_1)^{k_1} \cdots (\lambda-\lambda_r)^{k_r} \cdot q(\lambda)$

$q(\lambda) = 0$ has no solutions in $\mathbb{R}$.

$\lambda_1 \dots \lambda_r$ distinct eigenvalues of A.

r could be 0. q has degree $n - (k_1 + \cdots + k_r)$

k_i is the algebraic multeplicity of λ_i

$n_1 + \cdots + n_r \leq n$

Example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$p(\lambda) = -(\lambda-1)(\lambda^2+1)$

$\quad - (\lambda-1)(\lambda+i)(\lambda-i)$

Question: describe $T \; \mathbb{R}^3 \to \mathbb{R}^3$

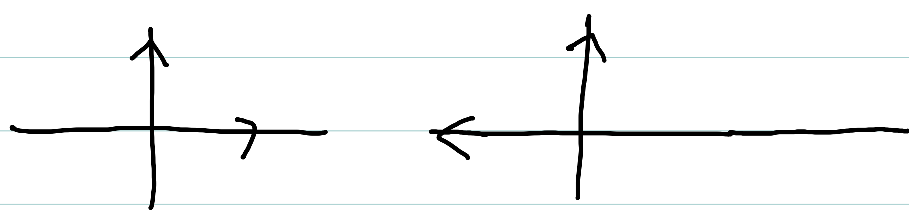
$T(v) = Av$

Can you find a 2x2 matrix with no eigenvalues? (In $\mathbb{R}$)

Geometric way: think of $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(v) = Av$. $T(v) = \lambda v$ means that
T "fixes" the line through origin
parallel to v.

Take for example T = rotation of $90°$
counterclockwise. What is the matrix of T?

$$A = \begin{bmatrix} T\begin{pmatrix}1\\0\end{pmatrix} & T\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



A should have no (real) eigenvalues:

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} =$$
$$= \lambda^2 + 1$$

$\lambda^2 + 1 = 0$ has no solutions in $\mathbb{R}$

Solutions in $\mathbb{C}$ are $-i, i$

Def: If $A \in R^{n\times n}$, λ_1 an eigenvalue of A then $N(A-\lambda_1 I)$ is called the Eigenspace of λ_1 and denoted by E_{λ_1}.

E_λ is a subspace of R^n

Every non zero vector in E_{λ_1} is an eigenvector of A for eigenvalue λ_1

$1 \le \dim E_{\lambda_1} \le$ algebraic multiplicity of λ_1

$\dim E_\lambda$ is called the geometric multiplicity of λ_1

Ex $A = \begin{bmatrix} 5 & 0 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ $p(\lambda) = -(\lambda-5)(\lambda-3)^2$

$E_5 = \text{Null} \begin{bmatrix} 0 & 0 & 0 \\ -4 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

solution in vector form $(2x_3, -4x_3, x_3)$

Basis for E_5 $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

Check $A \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -20 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$

$$E_3 = \text{Null}\begin{bmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $E_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Check $\begin{bmatrix} 5 & 0 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

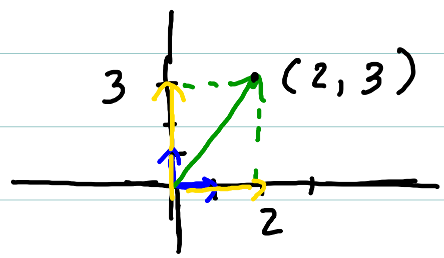
$$\begin{bmatrix} 5 & 0 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

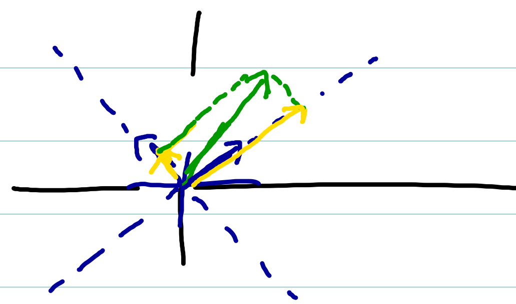
Is A diagonalizable ?

Change of basis

$$E = \underset{e_1}{\begin{bmatrix}1\\0\end{bmatrix}} \; \underset{e_2}{\begin{bmatrix}0\\1\end{bmatrix}} \quad \text{and} \quad \underset{b_1}{\begin{bmatrix}1\\1\end{bmatrix}} \; \underset{b_2}{\begin{bmatrix}-1\\1\end{bmatrix}} = B$$

are two bases in R^2





$$\begin{bmatrix}2\\3\end{bmatrix} = 2e_1 + 3e_2$$

$$\begin{bmatrix}2\\3\end{bmatrix} = \frac{5}{2} b_1 + \frac{1}{2} b_2$$

$$\left[\begin{bmatrix}2\\3\end{bmatrix} \right]_B = \begin{bmatrix}\frac{5}{2}\\ \frac{1}{2}\end{bmatrix}$$

Calculations to find $\left[\begin{bmatrix}2\\3\end{bmatrix}\right]_B$:

$$\left[\begin{array}{cc|c} \underbrace{\begin{matrix}1 & -1\\ 1 & 1\end{matrix}}_{B} & & \begin{matrix}2\\3\end{matrix}\end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & -1 & 2\\ 0 & 2 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2\\ 0 & 1 & 1/2\end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 5/2\\ 0 & 1 & 1/2\end{array}\right]$$

$$\left[\begin{bmatrix}2\\3\end{bmatrix}\right]_B = \begin{bmatrix}5/2\\ 1/2\end{bmatrix}$$

Def : the canonical basis for $\mathbb{R}^n$ is E : $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ .... $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Def: **Matrices of change of basis**

Given $E: e_1 \dots e_n$, and $B: b_1 \dots b_n$, where E is the canonical basis for $\mathbb{R}^n$, B is another basis, we can define matrices U_E^B and U_B^E s.t.:

$$U_E^B v = [v]_B \qquad U_B^E [v]_B = v$$

Recall:

$$U_B^E = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} \qquad U_E^B = (U_B^E)^{-1}$$

Ex $\quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$U_B^E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad U_E^B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Inverse calculation:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{pmatrix}$$

$$U_E^B = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \qquad U_E^B \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$$