Here are some extra practice problems. **You need to justify all your answers**.

1. Let $A \begin{pmatrix} 2 & 2 & -1 \\ 2 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

   i) Write $x^T Ax$ as a quadratic polynomial $Q(x_1, x_2, x_3)$.
   ii) Is A PSD or PD ? Justify your answer.

   Answer True or False, with justification:

   (a) Is A is symmetric then A^2 is PSD.
   (b) If A is PSD and B is PD, then A+kB is PSD for all $k \in \mathbb{R}$.

2. Suppose K_n is the complete graph on n vertices, that is K_n has n vertices and any two vertices i and j, with $i \neq j$ are connected by an edge. Let J_n be the $n \times n$ matrix with all entries equal to 1 and I_n be the $n \times n$ identity matrix.

   (a) Argue that $L_{K_n} = nI_n - J_n$.
   (b) Find all eigenvalues, with multiplicity, of $-J_n$
   (c) Find all eigenvalues, with multiplicity, of L_{K_n}

3. Suppose A is a symmetric matrix such that $A^3 = 0$, show that $A = 0$.

4. Show that the set S of all polynomials $p(x)$ of degree at most 2 such that $p(1) = 0$ is a subspace of $\mathbb{R}[\,x]_{\leq 2}$ and and find a basis B for S. Remember to justify why B is a basis.

5. You are given two orthogonal (but not orthonormal) sets in R^3: $B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

   $B_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ Consider the linear transformation $T : R^3 \to R^3$ such that

   $$T(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$
   $$T(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$
   $$T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

   Let A be the matrix of T, that is $T(v) = Av$ for all $v \in \mathbb{R}^3$. Find a SVD for A.

6. Consider : $\mathbb{R}[\,x]_{\leq 2} \to \mathbb{R}[\,x]_{\leq 3}$ defined by $T(p(x)) = xp(x)$

   (a) Show that T is a linear transformation.
   (b) Find the matrix of T.

Problem 1

1) i) $Q(x_1, x_2, x_3) = 2x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3$

ii) We can write Q as

$$x_1^2 + 4x_1x_2 + 4x_2^2 + x_1^2 + x_3^2 - 2x_1x_3 =$$
$$= (x_1 + 2x_2)^2 + (x_1 - x_3)^2$$

PSD but not PD since $Q(2, -1, 2) = 0$

2) a) $A^2 = A^T A$ so it is PSD. TRUE

OR: $(AA)^T = A^T \cdot A^T = AA$ so A^2 is symmetric.

If the eigenvalues of A are $d_1 \cdots d_n$

and $A v_i = d_i v_i$ then $A A v_i = A d_i v_i = d_i^2 v_i$

so the eigenvalues of A^2 are $d_1^2, d_2^2, \dots d_n^2$

and they are all ≥ 0 so A^2 is PSD TRUE

b) False Take for example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $k = -2$ (The problem here is having $k < 0$)

Then $A + (-2)B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and this matrix has negative eigenvalues $-1, -2$ so it is not PSD

Problem 2

a) The degree of each vertex in K_n is $n-1$ and there is an edge $\{i,j\}$ for every $1 \leq i < j \leq n$ so L_{K_n} has $n-1$ along the diagonal and -1 everywhere else and therefore
$$L_{K_n} = nI - J_n .$$

b) $-J_n$ is symmetric, it has rank 1 and therefore nullity $= n-1$ so $GM(0) = AM(0) = n-1$. The only non zero eigenvalue must be $\lambda = -n$ since $\operatorname{trace}(-J_n) = -n$ and the trace is the sum of the eigenvalues. So the eigenvalues of $-J_n$ are $0 \dots 0 \; -n$

c) By b) $\det(-J_n - \lambda I_n) = 0$ has solutions $\lambda = 0 \dots 0 \; -n$;

$\det(nI_n - J_n - \lambda I_n) = 0 \iff$
$\det(-J_n - (\lambda - n)I_n) = 0$ and therefore the solutions are $\lambda - n = 0 \dots 0 \; -n$ or $\lambda = n \dots n \; 0$

Problem 3

Since A is symmetric A is orthogonally diagonalizable : $A = QDQ^T$ Therefore

$0 = A^3 = QD^3Q^T$ which means

$Q^T 0 Q = D^3$ so

$$D^3 = \begin{bmatrix} d_1^3 & 0 & 0 \\ 0 & d_2^3 & 0 \\ 0 & 0 & d_3^3 \end{bmatrix} = 0 \quad \text{so} \quad d_1 = d_2 = d_3 = 0$$

Then $A = Q\,0\,Q^T = 0$

Problem 4

We need to check that :

1) The 0 polynomial in $\mathbb{R}[x]_{\leq 2}$ is in S:

$0(1) = 0$ so $0 \in S$

2) If $p \in S$ and $k \in \mathbb{R}$ $kp \in S$:

$(kp)(1) = k\,p(1) = 0$ since $p \in S$, therefore $kp \in S$.

3) If $p \in S$ and $q \in S$ then $p+q \in S$:

$(p+q)(1) = p(1) + q(1) = 0$ since p and q are in S. Therefore $p+q \in S$.

If $p(x) = a + bx + cx^2$ $\quad p(1) = a + b + c$

if we identify the polynomial $p(x)$ with the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$

Then S is identified with the following subspace of $\mathbb{R}^3$:

$$T = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \quad a+b+c=0 \right\}$$

A basis for T is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

going back to polynomials we get

$p = 1-x \qquad q = x - x^2$

We can verify that $p(1) = q(1) = 0$

p and q are linearly independent:

Suppose $k(1-x) + h(x-x^2) = 0$ (the 0 polynomial)

then $k + (h-k)x - hx^2 = 0$ so $h = k = 0$

clearly $S \neq \mathbb{R}[x]_{\leq 2}$ so

S has dimension at most 2

Therefore $S = \text{span}(1-x, x-x^2)$

and $1-x$, $x-x^2$ are a basis for S

Problem 5

Let $B_1' = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{x}_1, \vec{x}_2, \vec{x}_3$

$B_2' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \vec{y}_1, \vec{y}_2, \vec{y}_3$

So we want $T[\vec{x}_1] = \frac{2}{\sqrt{2}} \vec{y}_1$

$T[\vec{x}_2] = \frac{\sqrt{5}}{\sqrt{2}} \vec{y}_2$

$T[\vec{x}_3] = \sqrt{5}\, \vec{y}_3$

Note that $\sqrt{5} > \frac{\sqrt{5}}{\sqrt{2}} > \frac{2}{\sqrt{2}}$

So Let $v_1 = \vec{x}_3 \quad v_2 = \vec{x}_2 \quad v_3 = \vec{x}_1$

$u_1 = \vec{y}_3 \quad u_2 = \vec{y}_2 \quad u_3 = \vec{y}_1$

The linear transformation T, with respect to $B_1' = v_1\ v_2\ v_3$ in the domain and $B_2' = u_1\ u_2\ u_3$ in the codomain is given by

$$\Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5}/\sqrt{2} & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{2}} & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

Problem 6

1) $\forall p \in R[x]_{\leq 2}$ $\forall k \in R$, $T(kp) = x(kp(x))$
$= k\, x\, p(x) = k\, T(p)$

2) $\forall p, q \in R[x]_{\leq 2}$ $T(p+q) = x(p+q)$
$= xp + xq = T(p) + T(q)$

Therefore T is a linear transformation

We look at the basis B_1: $1\ x\ x^2$ in $R[x]_{\leq 2}$ and
B_2: $1\ x\ x^2\ x^3$ in $R[x]_{\leq 3}$

The matrix of T is

$$M = \begin{bmatrix} [T(1)]_{B_2} & [T(x)]_{B_2} & [T(x^2)]_{B_2} \end{bmatrix}$$

$T(1) = x = 0 + 1\cdot x + 0x^2 + 0x^2$
$T(x) = x^2 = 0 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3$
$T(x^2) = x^3 = 0 + 0\cdot x + 0\cdot x^2 + 1\cdot x^3$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OR:

1 corresponds to the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$

x corresponds to $\begin{bmatrix}0\\1\\0\end{bmatrix}$

x^2 corresponds to $\begin{bmatrix}0\\0\\1\end{bmatrix}$

$T(1)=x$ corresponds to the vector $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$T(x)=x^2$ corresponds to $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

$T(x^2)=x^3$ corresponds to $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

So we want to find the matrix of

S $R^3 \to R^3$ s.t $S\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $S\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$,

$S\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

$$M=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Let' check how it works for polynomials :

Given $p = a + bx + cx^2$ in $\mathbb{R}[x]_{\leq 2}$

$T(p) = ax + bx^2 + cx^3$ in $\mathbb{R}[x]_{\leq 3}$

p corresponds to the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$

$T(p)$ corresponds to the vector $\begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^4$

and

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix}$$