Here are some extra practice problems. The actual final will be shorter. **You need to justify all your answers**.

1. Let $A \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

   (a) Write $x^T Ax$ as a quadratic polynomial.

   (b) Is A PSD or PD ? Justify your answer.

   Answer True or False, with justification:

   (a) If A is symmetric then A^2 is PD.

   (b) If A is PSD and B is PD, then A+kB is PSD for all $k \in \mathbb{R}$.

2. Suppose K_n is the complete graph on n vertices, that is

$$K_n =< V = \{1, 2, \cdots n\}, E = \{\{i, j\}, 1 \leq i < j \leq n\} >$$

   Let J_n be the $n \times n$ matrix with all entries equal to 1 and I_n be the $n \times n$ identity matrix.

   (a) Argue that $L_{K_n} = nI_n - J_n$.

   (b) Find all eigenvalues, with multiplicity, of $-J_n$

   (c) Find all eigenvalues, with multiplicity, of L_{K_n}

3. Suppose A is a symmetric matrix such that $A^3 = 0$, show that $A = 0$.

4. Show that the set S of all polynomials $p(x)$ of degree at most 3 such that $p(1) = 0$ is a subspace of $\mathbb{R}[x]_{\leq 3}$ and and find a basis B for S. Remember to justify why B is a basis.

5. You are given two orthogonal (but not orthonormal) sets in R^3: $B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

   $B_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ Consider the linear transformation $T : R^3 \to R^3$ such that

$$T(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

   Let A be the matrix of T, that is $T(v) = Av$ for all $v \in \mathbb{R}^3$. Find a SVD for A.

6. Consider : $\mathbb{R}[\,x]_{\leq 2} \to \mathbb{R}[\,x]_{\leq 3}$ defined by $T(p(x)) = xp(x)$.

   (a) Show that T is a linear transformation.

   (b) Find the matrix of T.

7. I have collected some data and stored it in the rows of matrix A. unfortunately I can no longer read some of the entries of A. I know that $A = \begin{bmatrix} 1 & ? & 3 & ? \\ ? & ? & -1 & 2 \\ ? & -1 & -2 & ? \end{bmatrix}$. My data is centered around the origin, that is every column of A adds up to 0. I have done some PCA analysis on my data and I remember that the first principal component is $w \begin{pmatrix} 0.265 \\ 0.515 \\ 0.765 \\ -0.28 \end{pmatrix}$
   Below is the plot of my data along the direction of the principal component w , where point P_i corresponds to the projection of the i$^{\text{th}}$ row of A. Can you help me retrieve all entries of A? (Note : you will need to do some computations for this problem, you can use a calculator or anything you like; there will not be any long computations to do in the final).