Winter 2023 Math 318 midterm

NAME (First,Last) : ....

Student ID ....

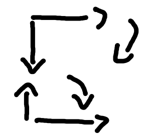
UW email ....

- Please use the same name that appears in Canvas.
- IMPORTANT: Your exam will be scanned: DO NOT write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of every non blank page of this exam.
- If you run out of space, continue your work on the back of the previous page and indicate clearly on the problem page that you have done so.
- Unless stated otherwise, you **MUST** justify your answers and show your work.
- Your work needs to be neat and legible.

Problem 1. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$



1. The linear transformation $T: R^2 \to R^2$, $T(v) = Av$ is a **clockwise** rotation of an angle θ: what is θ? $90°$

2. Does A have any real eigenvalues ? Justify your answer.

No because a rotation of $90°$ does not fix any lines.

OR

$p(\lambda) = \lambda^2 + 1$ $\quad$ $\lambda^2 + 1 = 0$ has no real solutions. A has no real eigenvalues.

3. Argue that if α is an eigenvalue for a matrix B, then α^2 is an eigenvalue for B^2.

Suppose $Bv = \alpha v$ for some $v \neq \vec{0}$

Then $B^2 v = B(Bv) = B\alpha v = \alpha Bv = \alpha^2 v$

Therefore α^2 is an eigenvalue for B^2

4. If we consider only real (not complex) eigenvalues, is it true that if B^2 has real eigenvalue μ , then B has real eigenvalue $\alpha = \sqrt{\mu}$? Explain.

I should really have asked : then must B have eigenvalues $\alpha = \sqrt{\mu}$ or $\alpha = -\sqrt{\mu}$?

As the question is phrased $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a counterexample since

B^2 has eigenvalue 1 and B does not have eigenvalue $\sqrt{1} = 1$ but what I

had in mind is the following example :

No for the matrix A of this problem

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ so A^2 has eigenvalues $-1, -1$

A has no real eigenvalues

($\sqrt{-1}$ is a complex number)

NAME (First,Last) :

Problem 2. A 1970 survey determined that 50% of American car owners drove large cars, 30% drove middle sized cars and 20 % drove small cars. A second survey in 1980 determined that 40% of large car owners in 1970, still drove a large car in 1980, but 10% switched to a middle sized car and 50% switched to a small car; of the middle sized car owners in 1970, 20% had switched to a big car, 70% were still driving a middle sized car, and 10% were owning a small car;finally, of the small car owners in 1970, 20% owned a large car in 1980, 20% owned a middle sized car, and 60 % still owned small cars.

1. Use a Markov matrix M to describe this situation, that is you want to find a matrix M, such that $M \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where a, b, c represent the percentages of car owners driving large, medium sized and small cars respectively in 1980.

$$M = \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{bmatrix}$$

2. $\begin{bmatrix} 0.5 \\ 0.7 \\ 0.8 \end{bmatrix}$ is an eigenvector for M. What is the eigenvalue associated with it ? Explain how you know.

$\lambda = 1$ because $\lambda = 1$ is the dominant eigenvalue of a positive Markov matrix and only the dominant eigenvalue has a positive eigenvector.

3. Assuming that every 10 years American switch cars as described above, determine the percentage of Americans that will own large, medium sized and small cars respectively in the long run.

We want $\lim_{k \to +\infty} M^k \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}$ we know this limit is v_1 = the eigenvector for $\lambda = 1$ that is a probability vector. $v_1 = \frac{1}{2} \begin{bmatrix} 0.5 \\ 0.7 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.35 \\ 0.4 \end{bmatrix}$

25% will drive big car, 35% will drive medium size car, 40% a small car

Problem 3. In a hw problem you proved that if P is the matrix of a projection, then P is diagonalizable and the only eigenvalues of P are λ =0 or 1. Prove that the converse is true, that is if A is a diagonalizable matrix whose only eigenvalues are 0 or 1, then A is the matrix of a projection, that is $A^2 = A$.

Assume $A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$, then $A^2 = P \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} P^{-1}$

since $\lambda_i = 0$ or 1 $\lambda_i^2 = \lambda_i$ so $A^2 = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1} = A$

Give an example of a matrix A having no eigenvalues different from 0 and 1 which is not the matrix of a projection.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ the eigenvalues of A are $\lambda = 1, 1$

you can verify that A is not diagonalizable

or compute $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A$

Let W be span ($\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$), and V be the plane x+y+z=0 . Calculate the orthogonal projections of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ on W and on V.

i) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) Method 1 : Find orthonormal basis for W : start with $b_1\ b_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ then

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

projection $u_1^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot u_1 + u_2^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot u_2$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + \frac{1}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 1/6 \\ -2/6 \end{bmatrix}$$

$$= \begin{bmatrix} 4/6 \\ -2/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

Method 2 using hw 5

$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is $\perp$ to the plane Π: $x + y + z = 0$

consider $u = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore the matrix of the orthogonal projection on Π is $P = I - u u^T =$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

Method 3 (Longest)

Find a basis for $\Pi : x+y+z=0$

$$B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix},$$ Projection is

$$P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & 1/3 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2/3 & 1/3 \\ 0 & 1 & 1/3 & 2/3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$