Lesson 8

Fibonacci's numbers

Hw 1 graded

Group hw 1 graded and reopened

Extracredit

$\forall K \geq 1 \quad P(K) \Rightarrow P(K+1)$

Assume K in $\mathbb{Z}$ $K \geq 1$, assume $P(K)$ ....

Recall the definition of Fibonacci's sequence:

$U_1 = 1$ want $\forall n \geq 1 \ P(n)$

$U_2 = 1$

$U_{n+1} = U_n + U_{n-1}$

Show $\forall n \geq 1$ $\underbrace{U_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}}_{P(n)}$, where

$\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ are the

solutions of $x^2 - x - 1 = 0$ so $\alpha^2 - \alpha - 1 = 0$
$\alpha^2 = \alpha + 1$

Proof by induction

1) Base cases:

$$\frac{\alpha - \beta}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1 = u_1$$

$$\frac{\alpha^2 - \beta^2}{\sqrt{5}} = \underbrace{\frac{\alpha - \beta}{\sqrt{5}}}_{=1} \cdot (\alpha + \beta) = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1 = U_2$$

2) Inductive step: assume $k \geq 2$, $k \in \mathbb{Z}$

$U_{k-1} = \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}}$, $U_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$ then

$$U_{k+1} = U_k + U_{k-1} = \frac{\alpha^k - \beta^k}{\sqrt{5}} + \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}} =$$

$$\frac{\alpha^{k-1}(\alpha+1) - \beta^{k-1}(\beta+1)}{\sqrt{5}}$$

since α, β are solutions of $x^2 - x - 1 = 0$

$\alpha^2 - \alpha - 1 = 0$ so $\alpha + 1 = \alpha^2$ and

$\beta^2 - \beta - 1 = 0$ so $\beta + 1 = \beta^2$ so

$$U_{k+1} = \frac{\alpha^{k-1} \cdot \alpha^2 - \beta^{k-1} \beta^2}{\sqrt{5}} = \frac{\alpha^{k+1} - \beta^{k+1}}{\sqrt{5}}$$

So we proved $\forall k \geq 2$ $P(k-1) \wedge P(k) \Rightarrow P(k+1)$

where $P(k)$ says $U_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}$

Inductive step proves:

$P(1) \wedge P(2) \Rightarrow P(3)$

$P(2) \wedge P(3) \Rightarrow P(4)$

$P(3) \wedge P(4) \Rightarrow P(5)$

$\vdots$

$\alpha = \frac{1+\sqrt{5}}{2}$ is called GOLDEN RATIO

Induction recep:

Reguler induction

1) Base case: proue $P(n_0)$

2) Inductive step: proue $\forall k \geq n_0 \quad P(k) \Rightarrow P(k+1)$

$$\left.\begin{array}{l} P(n_0) \\ P(n_0) \Rightarrow P(n_0+1) \\ \vdots \\ P(k) \Rightarrow P(k+1) \end{array}\right\} \text{are true}$$

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Conclude $\forall n \geq n_0 \quad P(n_0)$ is true

Strong Induction

1) Base case: proue $P(n_0)$

2) Inductive step: $\forall k \geq n_0 \quad (P(n_0) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1)$

$$\left.\begin{array}{l} P(n_0) \\ P(n_0) \Rightarrow P(n_0+1) \\ \vdots \\ P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1) \end{array}\right\} \text{are true}$$

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Conclude $\forall n \geq n_0 \quad P(n_0)$ is true but some problems require an inductive step of the form:

$$\forall k \ge n_0+1 \;\; p(k-1) \wedge p(k) \Rightarrow p(k+1)$$

What base case do we need to prove $\forall n \ge n_0 \;\; P(n)$?

$P(n_0)$, $P(n_0+1)$ base cases

$$P(n_0) \wedge p(n_0+1) \Rightarrow p(n_0+2)$$
$$P(n_0+1) \wedge p(n_0+2) \Rightarrow p(n_0+3)$$
$$\vdots$$
$$p(k-1) \wedge p(k) \Rightarrow p(k+1)$$
$$\vdots$$

Modified induction

To prove $\forall n \geq n_0 \quad P(n)$

Proove

1) Base cases: $\overbrace{P(n_0),\ P(n_0+1) \cdots P(n_0+s-1)}^{s \text{ base cases}}$

2) Inductive step: Prove

$$\forall k \geq n_0+s-1 \quad \underbrace{P(k-s+1) \wedge \cdots \wedge P(k-1) \wedge P(k)}_{s \text{ premises}} \Rightarrow P(k+1)$$

How does it work?

$$\left.\begin{array}{l} P(n_0) \quad P(n_0+1) \cdots\cdots \quad P(n_0+s-1) \\ P(n_0) \wedge \cdots\cdots \wedge P(n_0+s-1) \Rightarrow P(n_0+s) \\ P(n_0+1) \wedge \cdots \wedge P(n_0+s) \Rightarrow P(n_0+s+1) \\ \vdots \\ P(k-s+1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1) \\ \cdots\cdots \end{array}\right\} \text{are True}$$

Then $\forall n \geq n_0 \quad P(n)$ is true

Define

$a_1 = 1$

$a_2 = 9$

$a_{n+1} = 6a_n - 9a_{n-1}$ for $n+1 \geq 3$

Show $a_n = (2n-1)\,3^{n-1}$: $P(n)$

Want $\forall n \geq 1$ $P(n)$

Proof by induction:

1) Base case: $(2\cdot 1 - 1)\cdot 3^{1-1} = 1 = a_1$ $P(1)$

$(2\cdot 2-1)\,3^{2-1} = 9 = a_2$ $P(2)$

2) Inductive step: assume $k \geq 2$, k in $\mathbb{Z}$

$a_{k-1} = (2(k-1)-1)\cdot 3^{(k-1)-1}$ and

$a_k = (2k-1)\,3^{k-1}$

Then $a_{k+1} = 6\,(2k-1)\,3^{k-1} - 9\,(2(k-1)-1)\,3^{k-2}$

$= 2(2k-1)\cdot 3^k - (2k-3)\,3^k =$

$= (4k-2-2k+3)\,3^k =$

$= (2k+1)\,3^{k+1-1} =$

$= (2(k+1)-1)\,3^{k+1-1}$

U_n is the nth Fibonacci

Th $\forall n \geq 1 \quad \sum_{i=1}^{n} U_{2i} = U_{2n+1} - 1$

$U_2 + U_4 + \cdots + U_{2n} = U_{2n+1} - 1$

Proof by induction

1) Base case : recall $U_1 = 1,\ U_2 = 1,\ U_3 = 2$

so, if $n = 1 \quad \sum_{i=1}^{1} U_{2i} = U_2 = 1 = U_3 - 1$

2) Inductive step: assume $k \geq 1$ and

$\sum_{i=1}^{k} U_{2i} = U_{2k+1} - 1$, then

$\sum_{i=1}^{k+1} U_{2i} = \sum_{i=1}^{k} U_{2i} + U_{2(k+1)} = U_{2k+1} - 1 + U_{2k+2} =$

$= U_{2k+3} - 1 = U_{2(k+1)+1} - 1$