Lesson 7

Strong induction
Recursive definitions

$\boxed{\text{Strong induction}}$ is used to prove statements of the form

$\forall n \geq n_0 \quad P(n)$ (we assume n in $\mathbb{Z}$)

Strong induction has 2 steps:

1) Base case: prove $P(n_0)$
2) Inductive step: prove

$$\forall k \geq n_0 \quad (P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1))$$

Why does it work?

$$\left.\begin{array}{l} P(n_0) \\ P(n_0) \Rightarrow P(n_0+1) \\ P(n_0) \wedge P(n_0+1) \Rightarrow P(n_0+2) \\ P(n_0) \wedge P(n_0+1) \wedge P(n_0+2) \Rightarrow P(n_0+3) \end{array}\right\} \text{True}$$

....

Allow us to conclude

$P(n_0),\ P(n_0+1),\ P(n_0+2),\ P(n_0+3) \ldots$

Def : An integer n is a prime if
1) $n > 1$ (So 1 is NOT a prime)
2) the only positive divisors of n are 1 and n

An integer is composite if it is not prime.

Th : Every integer greater than 1 can be written as the product of 1 or more Primes.

Proof : let $P(n)$ = n can be written as the product of 1 or more primes. So we need to prove $\forall n \geq 2 \; P(n)$

We will do a proof by strong induction.

1) Base case : $2 = 2$, the product of one prime.
2) Inductive step : assume $k \geq 2$ $k \in \mathbb{Z}$ and that $2, 3, \ldots\ldots, k$ can be written as product of primes. Consider $k+1$; either $k+1$ is prime, in which case we are done, or $k+1$ is divisible by a, with $2 \leq a \leq k$ so $k+1 = a \cdot b$ and b is also between 2 and k, so by induction assumption both a and b can be written as product of primes, and therefore $k+1$ can be written

as product of primes.

Note : from now on you can use the unique factorization th : every integer greater than one can be written as product of primes, in a unique way (up to reordering of the factors)

although we have not proved the uniqueness part yet.

Def : A sequence of numbers $\{a_n\}$ is an infinite list of numbers that are indexed by the natural numbers: $a_1, a_2, a_3 \dots$

$\{a_n\}$ can be defined explicitly : ex : $a_n = n^2$
$\{a_n\}$ is $1, 4, 9, 16, \dots$

OR

$\{b_n\}$ can be defined by recursion:

ex: $b_1 = 1$

$b_{n+1} = (n+1) \cdot b_n$ for $n+1 \geq 2$

What is b_5?

$b_5 = 5\, b_4 = 5 \cdot 4 \cdot b_3 = 5 \cdot 4 \cdot 3 \cdot b_2 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot b_1 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$b_n = n!$

Th: $\forall n \geq 4$ $n! > 2^n$ $P(n)$

Proof by induction

1) Base case: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$2^4 = 16$

therefore $4! > 2^4$

$P(k)$

2) Inductive step: assume $k \geq 4$, k in $\mathbb{Z}$ and $k! > 2^k$, then

$(k+1)! = (k+1)\, k! > (k+1) \cdot 2^k > 2 \cdot 2^k = 2^{k+1}$

induction assumption $k! > 2^k$

$k+1 > 2$ since $k \geq 4$

therefore $(k+1)! > 2^{k+1}$

Fibonacci's sequence.

$U_1 = 1$

$U_2 = 1$

$U_{n+1} = U_n + U_{n-1}$ if $n \geq 2$

$U_{K+1} = U_K + U_{K-1}$

What is U_5 ?

$U_5 = U_4 + U_3 = 3 + 2 = \boxed{5}$

$U_4 = U_3 + U_2 = 2 + 1 = 3$

$U_3 = U_2 + U_1 = 1 + 1 = 2$

Th : $\forall n \geq 1$ $\overbrace{U_n \text{ is defined}}^{P(n)}$

Proof : by strong induction

1) Base case : $U_1 = 1$ so U_1 is defined

2) Inductive step : assume $U_1\ U_2 \dots U_K$ are defined

then $U_{K+1} = U_K + U_{K-1}$ is defined.

This proof is wrong. Why ?

1) Base case : $U_1 = 1$ so U_1 is defined
$U_2 = 1$ so U_2 is defined

2) Inductive step : assume $U_1\ U_2 \cdots U_K$ are defined
then $U_{K+1} = U_K + U_{K-1}$ is defined.

U_1 is defined , U_2 is defined

$$\overset{P(1)}{} \qquad \overset{=>}{} \quad \overset{P(2)}{}$$

* ~~U_1 is defined => U_2 is defined~~

U_1 defined $\wedge$ U_2 defined => U_3 defined

U_1 def $\wedge$ U_2 def $\wedge$ U_3 def => U_4 def ...

⋮

$U_2 = U_1 + U_0$ Not true