Lesson 6

More examples of induction

Th: $\forall n \geq 1 \quad n \leq 2^n$

Proof by induction:

1) Base case: when $n=1$ $\quad 1 \leq 2^1$, clearly true.

2) Inductive step: assume $k \geq 1$, k in $\mathbb{Z}$ and $k \leq 2^k$, then $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot k \geq k+1$

Summation formulas

$$\underbrace{\sum_{i=1}^{n} i}_{\text{This is a number not a statement}} \quad \text{means} \quad 1+2+3+\cdots+n$$

$$\underbrace{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}_{\text{This is a predicate } P(n)}$$

(in the video I also call it an open statement)

$$\underbrace{\forall n \geq 1 \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}}_{\text{This is a statement}}$$

Th : $\forall n \geq 1 \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof : by induction on n

1) Base case : if $n = 1 \quad \sum_{i=1}^{1} i = 1 \quad (*)$

$\frac{1(1+1)}{2} = 1 \quad (**)$. Clearly

$(*) = (**)$

2) Inductive step : assume $k \geq 1$, k in $\mathbb{Z}$

and $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, then

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) =$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+1+1)}{2} .$$

We can generalize induction to prove statements of the form

$$\forall n \geq n_0 \quad P(n)$$

(again we implicitly assume n in $\mathbb{Z}$)

In this case the two steps of a proof by induction are:

1) Base case: prove $P(n_0)$
2) Inductive step: prove
   $\forall k \geq n_0 \quad P(k) \Rightarrow P(k+1)$

Th: $\forall n \geq 4 \quad n^2 \leq 2^n$

Proof: by induction

1) Base case: if $n=4$, clearly
$16 = 4^2 \leq 2^4 = 16$

2) Inductive step: assume $k \geq 4$,
k in $\mathbb{Z}$ and $k^2 \leq 2^k$, then

$2^{k+1} = 2 \cdot 2^k \geq 2k^2$

Now I want to prove that
$2k^2 \geq (k+1)^2 = k^2 + 2k + 1$ This is equivalent to $k^2 \geq 2k+1$, which is true because, since $k \geq 4$, we have $k^2 \geq 4k = 2k + 2k \geq 2k+1$

(This argument is different from the one given in the video)

Therefore $2^{k+1} \geq 2k^2 \geq (k+1)^2$

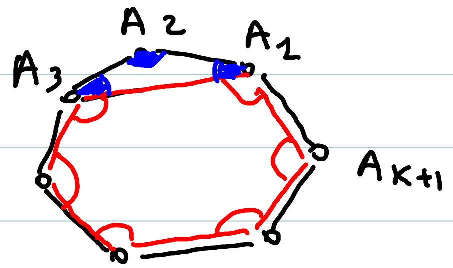
Th : The sum of the internal angles of a regular polygon with n sides is $180(n-2)$.
Call this statement $P(n)$. We want to prove $\forall n \geq 3 \; P(n)$

Proof: by induction

1) Base case : for $n=3$ our polygon is a triangle.

the sum of the angles of a triangle is $180\cdot(3-2)$. We will assume this (without giving a proof), from geometry

2) Inductive step: assume $k \geq 3$, k in $\mathbb{Z}$ and $P(k)$ is true.
Consider a polygon with $k+1$ sides



Draw an edge joining vertices A_1 and A_3.

The (red) polygon with vertices A_1 A_3 A_4 .... A_{k+1} has k vertices so the sum of its angles is $180(k-2)$. The sum of the internal angles of the original polygon is $180(k-2)+180 = 180(k+1-2)$