Lesson 5

Proofs by contradiction

Start induction

Grade the following proof

6 divides n : $\exists k$ in $\mathbb{Z}$ $n = 6k$

6 does not divide n : $\neg \exists k$ in $\mathbb{Z}$ $n = 6k$

$\forall k$ in $\mathbb{Z}$ $n \neq 6k$

Goal : 2 does not divide n $\vee$ 3 does not divide n

Th: $\forall n$ in Z (((2 div n) $\wedge$ (3 div n)) $\Rightarrow$ 6 div n):

Proof: by contraposition. Assume n is an integer

(1) Assume 6 does not divide n,

(2) this means $\forall k$ in $\mathbb{Z}$ $n \neq 6k$ therefore

(3) $\forall k$ in $\mathbb{Z}$ $n \neq 2(3k)$ and also $\forall k$ in $\mathbb{Z}$ $n \neq 3(2k)$

(4) therefore 2 and 3 do not divide n. ✗

2 does not divide n : $\forall h$ $n \neq 2h$; $\forall k$ $n \neq 2\,3k$) NOT EQUIVALENT

Proofs by contradiction:

To prove S, assume $\neg S$, derive a contradiction i.e a false statement.

$$\neg S \Rightarrow \text{False} \quad \text{is true} \quad \text{so}$$
$$\equiv$$
$$\text{False} \vee \neg(\neg S) \quad \text{is true}$$

So S is true.

In particular if S is an implication: $P \Rightarrow Q$, a proof of contradiction of S starts assuming P and $\neg Q$ and aims at deriving a false statement.

Can you try a proof by contradiction of

$\forall n$ in $\mathbb{Z}$

$\neg(($2 divides $n) \wedge ($3 divides $n) \Rightarrow$ 6 divides n :

Assume 2 divides n and 3 divides n and 6 does not divide n. Plan: derive a false statement.

We know $n = 2k$ and $n = 3h$ for some k and h in $\mathbb{Z}$, then $2k = 3h$, therefore h must be even, so $n = 3h = 3 \cdot 2t$ for some t in $\mathbb{Z}$

so 6 divides n and given the assumption that 6 does not divide n we conclude that

6 divides $n \wedge$ 6 does not divide n, which is clearly false.

Th: $\sqrt{2}$ is irrational.

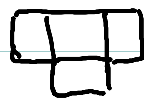
Proof by contradiction: assume $\sqrt{2}$ is not irrational, then $\sqrt{2} = \frac{a}{b}$ for some integers a, $b \neq 0$ and we can assume a and b have no common divisors other than 1 (i.e we can assume we have simplified our fraction).

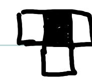
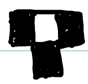
. Our goal now is to derive a contradiction.

$\sqrt{2} = \frac{a}{b}$ implies $\sqrt{2}\, b = a$ and $2b^2 = a^2$ (*) and therefore a^2 is even.

We have proved in Lesson 4 that a^2 is even implies a is even, so $a = 2k$ for some k in $\mathbb{Z}$. Plugging $a = 2k$ in (*) we get $2b^2 = (2k)^2$ so $2b^2 = 4k^2$ and therefore $b^2 = 2k^2$ which allows us to conclude that b is even.

Therefore a and b have 2 as common divisor, contradicting the assumption that a and b do not have a common divisor greater than 1.

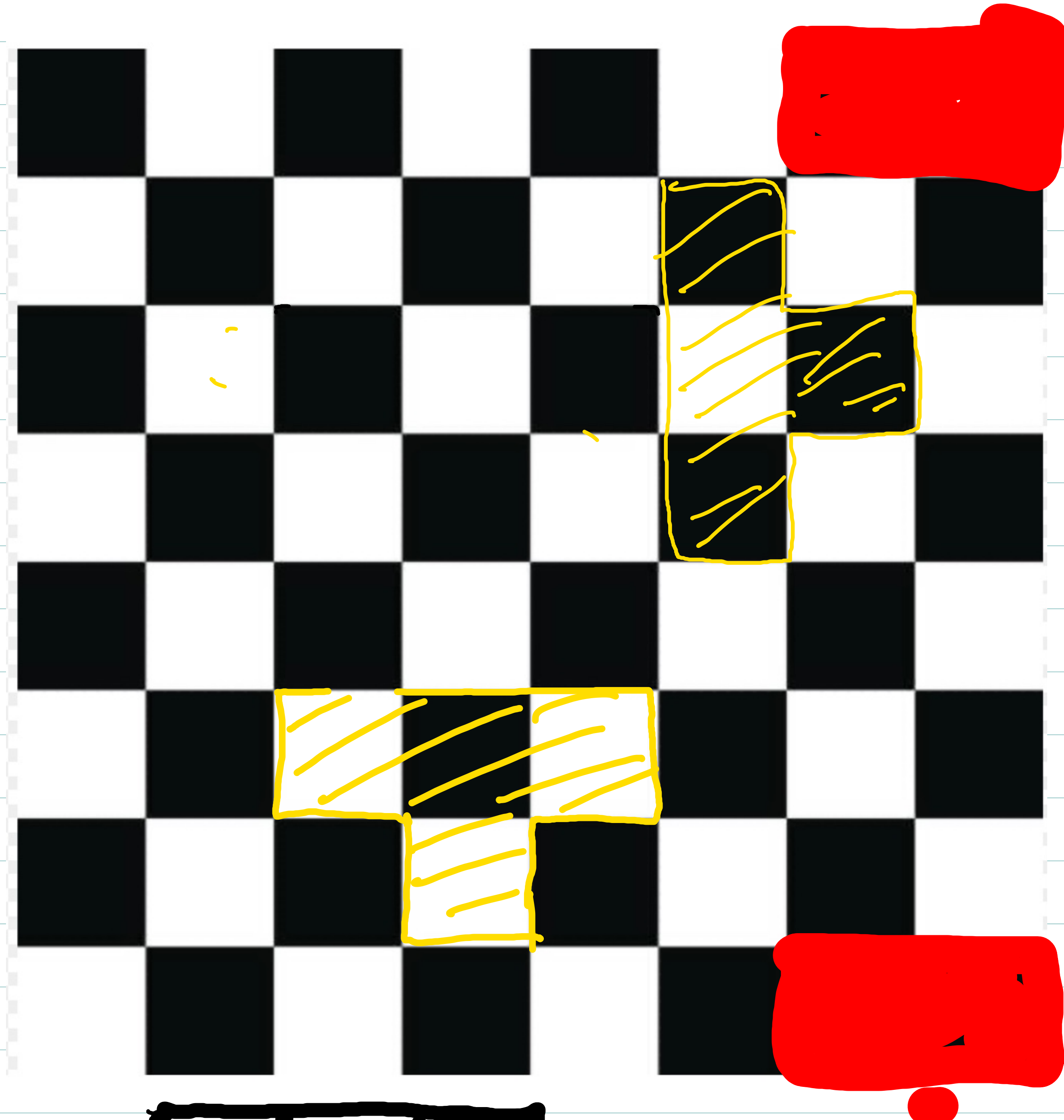
Th: A regular chessboard with the top 2 right squares and the bottom 2 right squares removed cannot be covered by tiles of this form  (T tile).

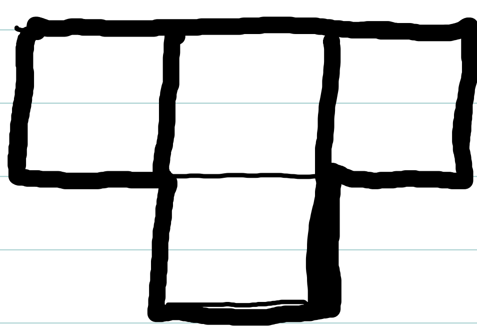
Proof: assume by contradiction that we can cover a (modified) chessboard with T tiles. We can color each tile either  (type 1) or  (type 2) in such a way that every black square on a T tile covers a black square on the board and every white square covers a white square on the board.
Suppose our covering uses m tiles of type 1 and n tiles of type 2. m and n are in $\mathbb{N}$ (integers ≥ 0)
Our board has 30 white squares and 30 black squares, therefore we must have:

$3m + n = 30$ (# white squares covered)
$m + 3n = 30$ (# black squares covered)

Solving for m and n gives $m = n = \frac{15}{2}$
so m and n are not integers, giving a contradiction.

IDEA: using colors may be useful in tiling problems.





T tile

Induction :

It is a proof technique used to prove statements of the form

$$\forall n \text{ in } \mathbb{Z}^+ \quad P(n)$$

We will also write

$\forall n \geq 1 \quad P(n)$ (assuming n is in $\mathbb{Z}$)

Generalizations later.

We need to prove $P(1), P(2), \ldots P(n), \ldots$

A proof by induction has 2 steps:

1) Prove $P(1)$ Base case
2) Prove $\forall k$ in $\mathbb{Z}^+$ $P(k) \Rightarrow P(k+1)$ Inductive step

Why does it work? We prove

$$\left.\begin{array}{l} T\ P(1) \\ T\ P(1) \Rightarrow P(2)\ )\ T \\ T\ P(2) \Rightarrow P(3)\ \ T \\ T\ P(3) \Rightarrow P(4)\ )\ T \\ \vdots \end{array}\right\} \text{ are all true So}$$

P(1) is true, P(2) is true, P(3) is true, ...

Th: $\forall n \geq 1$ $\underbrace{n^3 + 2n \text{ is divisible by } 3}_{P(n)}$

Proof by induction:

1) Base case: we need to prove $P(1)$.

if $n = 1$, $n^3 + 2n = 3$, which is clearly divisible by 3, so $P(1)$ is True

2) Inductive step: we need to prove $\forall k \geq 1$ $P(k) \Rightarrow P(k+1)$.

Assume $k \geq 1$, k in $\mathbb{Z}$ and $k^3 + 2k$ is divisible by 3, then $k^3 + 2k = 3h$ for some h in $\mathbb{Z}$, and

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (k^3 + 2k) + 3(k^2 + k + 1) = 3(h + k^2 + k + 1)$$

is also divisible by 3.