Lesson 3

Truth tables

Negation rules

3 proofs methods

E = Tom exercises
W = Tom puts on weight

The negation of "if Tom does not exercise he puts on weight " is (mark all that apply):

$$\neg(\neg E \Rightarrow W) \equiv \neg E \wedge \neg W$$

1. If Tom puts on weight he does not exercise.

$W \Rightarrow \neg E$ not equivalent look at: $W = F$, $E = T$

2. Tom does not exercise and does not put on weight.

$\neg E \wedge \neg W$

3. If Tom exercise he does not put on weight.

$E \Rightarrow \neg W$ not equivalent look at: $W = T$, $E = F$

4. Tom exercises and does not put on weight.

$E \wedge \neg W$ not equivalent look at: $W = F$, $E = F$

Negation rules (see handout)

Last time we looked at negation rules for Connectives.

Quantifiers negation rules

$\neg \forall x \; P(x)$ same as $\exists x \; \neg P(x)$

$\neg \exists x \; P(x)$ same as $\forall x \; \neg P(x)$

Negate

$$\exists \ell \text{ in } \mathbb{R} \ \ \forall \varepsilon > 0 \ (\varepsilon \text{ in } \mathbb{R}^+) \ \ \exists m \text{ in } \mathbb{Z}^+ \ \ \forall n \text{ in } \mathbb{Z}^+ \ \ n > m \Rightarrow |(-1)^n - \ell| < \varepsilon$$

and simplify the negation into a statement that does not use the symbol $\neg$

(If you take math 327 you will learn that the statement above says:

$$\lim_{n \to +\infty} (-1)^n \text{ exists} \quad)$$

Negation:

$$\neg \exists \ell \text{ in } \mathbb{R} \ \ \forall \varepsilon > 0 \ \ \exists m \text{ in } \mathbb{Z}^+ \ \ \forall n \text{ in } \mathbb{Z}^+ \ n > m \Rightarrow |(-1)^n - \ell| < \varepsilon$$

$$\forall \ell \text{ in } \mathbb{R} \ \ \exists \varepsilon > 0 \ \ \forall m \text{ in } \mathbb{Z}^+ \ \ \exists n \text{ in } \mathbb{Z}^+ \ \ (n > m) \wedge ((-1)^n - \ell) \geq \varepsilon$$

More terminology

If $P \Rightarrow Q$ is True (ASIDE: does this tell us anything about P or Q being True or False?) we say

P is sufficient for Q
Q is necessary for P

If $P \Leftrightarrow Q$ is true we say

P (Q) is necessary and sufficient for Q (P)

Is being a multiple of 4 a necessary condition for being even? Is it sufficient?

Consider

$\forall x \quad x$ is even $\Rightarrow x$ is a multiple of 4

Not true so being a multiple of 4 is not necessary for being even

Consider

$\forall x \quad x$ is a multiple of 4 $\Rightarrow x$ is even

True so being a multiple of 4 is sufficient for being even

Proofs and quentifiers

To prove $\forall x$ in A $P(x)$
Consider a generic x in A and show $P(x)$

To prove $\exists x$ in A $P(x)$
Find one element a of A and show $P(a)$

Three proof methods to prove

$\forall x \quad P(x) \Rightarrow Q(x)$

1) Direct method:

assume $P(x)$ derive $Q(x)$

2) Contraposition:

assume $\neg Q(x)$ derive $\neg P(x)$

3) Contradiction:

assume $P(x)$ and $\neg Q(x)$ and

derive a false statement.

In general to prove S by contradiction

assume $\neg S$ and derive a false statement.

Def: a in $\mathbb{Z}$ is even iff $\exists k$ in $\mathbb{Z}$ $a = 2k$

a in $\mathbb{Z}$ is odd iff $\exists k$ in $\mathbb{Z}$ $a = 2k+1$

Given a, b in $\mathbb{Z}$ a divides b iff $\exists k$ in $\mathbb{Z}$ $b = a \cdot k$.

Th: Being a multiple of 4 is sufficient for being even. This means

$$\forall x \text{ in } \mathbb{Z} \quad \underbrace{x \text{ is a multiple of } 4}_{P(x)} \Rightarrow \underbrace{x \text{ is even}}_{Q(x)}$$

is True

Proof: suppose x is a multiple of 4, that is $x = 4k$ for some k in $\mathbb{Z}$, then $x = 2(2k)$ and $2k$ is in $\mathbb{Z}$ therefore x is even

This is en exemple of a direct proof.

Th: Being a multiple of 4 is not necessary for being even. This means

$\neg \forall x \text{ in } \mathbb{Z} \, (x \text{ is even} \Rightarrow x \text{ is a multiple of } 4)$

is True.

Proof: the statement we need to prove is equivalent to:

$\exists x \text{ in } \mathbb{Z} \quad x \text{ is even} \wedge x \text{ is not a multiple of } 4.$

Take $x = 2$.

Note: to prove S is False we usually prove $\neg S$ is True.