Lesson 24

Equivalence relations and
Equivalence classes

From lesson 23 video:

$$f \quad \mathbb{Z}_m \longrightarrow \mathbb{Z}_m$$
$$f(x) = ax$$

What do we know about this function?

Def : a relation R on a set $A \neq \emptyset$ is a subset of $A \times A$

Ex : A : Math 300 students

$x R_1 y$ iff x and y have same GPA

$x R_2 y$ iff x and y belong to the same ethnic group

Def : A relation R on A is

1) Reflexive (r) if aRa for all $a \in A$

2) Symmetric (S) if $aRb \Rightarrow bRa$ for all $a, b \in A$

3) Transitive (t) if $aRb \wedge bRc \Rightarrow aRc$ for all $a, b, c \in A$

Def : a relation R on A that is reflexive, symmetric and transitive is an equivalence relation.

Ex : $=$ is an equivalence relation on any $A \neq \emptyset$

$\equiv_m$ is an equivalence relation on $\mathbb{Z}$

$<$ is a relation on $\mathbb{Z}$ but it is

not r, not s, t

$\leq$ is r, not s, t

Def : Given an equivalence relation R on A and $x \in A$, the equivalence class of x, $[x]_R$ is the set $\{a \in A \mid x R a\}$ $= \{a \in R \mid a R x\}$

Th : $\forall a, b \in R$ either $\underbrace{[a]_R = [b]_R}_{A}$ or $\underbrace{[a]_R \cap [b]_R = \emptyset}_{B}$

want to prove $A \vee B$. Prove $\neg B \Rightarrow A$ instead

Proof : Suppose $x \in [a]_R \cap [b]_R$

and $y \in [a]_R$, then $y R a$ and $a R x$

and $x R b$ so $y R b$ so $y \in [b]_R$

therefore $[a]_R \subseteq [b]_R$. In a similar way you can prove $[b]_R \subseteq [a]_R$

Therefore $[a]_R = [b]_R$

Th: $\forall a, b \in A \quad aRb \Leftrightarrow [a]_R = [b]_R$

Suppose aRb then $b \in [a]_R \cap [b]_R$

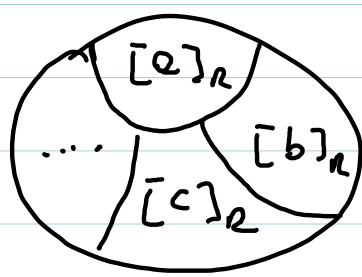
so $[a]_R = [b]_R$

Suppose $[a]_R = [b]_R$. $a \in [a]_R$ since

R is reflexive and therefore $a \in [b]_R$

so aRb

Th $\forall a, b \in A \quad a \not R b$ then $[a]_R \cap [b]_R = \emptyset$



Ex : $A = \mathbb{Z}$ $\quad R$ is $=$

describe equivalence classes

$[x]_R = \{x\}$

Ex $\quad A = \mathbb{Z}$ $\quad R$ is $\equiv_m$

describe equivalence classes

$[a]_R \equiv [a]_m$

Ex : Given a set U $\quad A = P(U)$

$x R y \iff |x| = |y|$ is an equivalence relation

1) (r) $|x| = |x|$ ie there is a bijection f $x \to x$

2) (s) if $|x| = |y|$ then $|y| = |x|$ if there is a bijection f $x \to y$ then there is a bijection g $y \to x$

3) if $|x| = |y|$ and $|y| = |z|$ then $|x| = |z|$

if there is a bijection f $x \to y$ and a bijection g $y \to z$ then there is a bijection h $x \to z$

Suppose $\mathbb{Z}^+ \subseteq U$ $\quad$ describe $[\mathbb{Z}^+]_R$

Ex: $A = \mathbb{Z} \times \mathbb{Z} - \{0\}$

$(a_1, b_1) \, R \, (a_2, b_2) \iff a_1 b_2 = a_2 b_1$

Is R an equivalence relation?

(r) $(a, b) \, R \, (a, b)$ since $ab = ab$

(s) suppose $(a, b) \, R \, (c, d)$ then $ad = bc$

so $bc = ad$ so $(c, d) \, R \, (a, b)$

(t) suppose $(a, b) \, R \, (c, d)$ and

$(c, d) \, R \, (e, f)$ then

$ad = bc$ and $cf = de$

so $a\cancel{d}\cancel{c}f = b\cancel{c}\cancel{d}e$ so $(a, b) \, R \, (e, f)$

describe equivalence classes

$$\frac{a}{b} \, R \, \frac{c}{d} \iff ad = bc$$

Equivalence classes of R are $\mathbb{Q}$

Functions on equivalence classes

$f \quad \mathbb{Z}_5 \longrightarrow \mathbb{Z}_5$

$$f([a]_5) = [a+1]_5$$

well defined since if $b \in [a]_5$
then $a+1 \equiv_5 b+1$

$g \quad \mathbb{Z}_5 \longrightarrow \mathbb{Z}$

$$g([a]_5) = a+1$$

nonsense what is $g[0]_5$? $\begin{matrix} 0+1 \\ 5+1 \\ -5+1 \end{matrix}$?