Lesson 17

More cardinality proofs examples

Is $[1,3]$ denumerable? No

1) Proof 1: repeat Cantor's argument
2) Proof 2: find a bijection between a set we know is not denumerable and $[1\ 3]$:

$$f\ [0\ 1] \longrightarrow [1\ 3]$$
$$f(x) = 2x + 1$$

is a bijection: you can check this yourself.
Assume by contradiction that $[1\ 3]$ was denumerable, then there would be a bijection $g\ [1\ 3] \to \mathbb{Z}^+$ and $g \circ f\ [0,1] \longrightarrow \mathbb{Z}^+$ would be a bijection. Impossible.

Is $\mathbb{Q}^+$, the set of positive fractions denumerable?

What is a rational number? $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} \dots$

When we write $\frac{a}{b}$ let's assume $\frac{a}{b}$ is reduced

Consider $f \quad \mathbb{Q}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$

$$f\left(\frac{a}{b}\right) = (a, b)$$

Clearly injective

Is it surjective? No for example $(2, 4)$ is not in the range of f.

$\mathbb{Q}^+$ is equipotent to an infinite subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$, so it is denumerable

Informal proof that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is denumerable

(1 1)
(1 2) (2 1)
(1 3) (2 2) (3 1)
(1 4) (2 3) (3 2) (4 1)
(1 5) (2 4) (3 3) (4 2) (5 1)
. . . . .

Is the set of irrational numbers denumerable? NO

$\mathbb{R} = \mathbb{Q} \cup$ Irrational

In hw: if A and B are denumerable then $A \cup B$ is denumerable

Are there sets of size bigger than $\mathbb{Z}^+$ and smaller than $\mathbb{R}$?
Continuum hypothesis.

Are there sets of size bigger than $\mathbb{R}$? Yes $P(\mathbb{R})$ (see worksheet 4)