Lesson 16

Denumerability proofs.

Q & A on Wednesday
Midterm covers Lessons 1 - 14
No cardinality.

Recall :

$|A| = |B|$ iff there is f $A \to B$ bijection

$|A| < |B|$ iff There is g $A \to B$ injective, but we cannot define a bijection $A \to B$

$|A| \leq |B|$ if there is g $A \to B$ injective

Note : $|A|$, $|B|$ may not be numbers !

$|\mathbb{Z}| = |\mathbb{Z}^+|$ $\qquad$ $|\mathbb{Z}^+| < |\mathbb{R}|$

so we are using $=$ $<$ $\leq$ with a different meaning then the usual equality and inequality between numbers

: a set A is denumerable if $|A| = |\mathbb{Z}^+|$

$[0,1]$, $\mathbb{R}$ are not denumerable.

Denumerable sets

$\mathbb{Z}^+$, $\mathbb{Z}$, ODD, EVEN, any infinite subset of a denumerable set, $\mathbb{Z}^+ \times \mathbb{Z}^+$ (hw 5)

Th $EVEN_{\mathbb{Z}^+} = \{x \in \mathbb{Z}^+ / x \text{ is even}\}$

is denumerable

Proof: define

$$f: EVEN_{\mathbb{Z}^+} \to ODD_{\mathbb{Z}^+}$$
$$f(n) = n-1$$

f is a bijection. Easy to check you can check it.

We defined in a previous lecture a bijection g $\mathbb{Z}^+ \to ODD_{\mathbb{Z}^+}$ so g^{-1} $ODD_{\mathbb{Z}^+} \to \mathbb{Z}^+$ is a bijection

and $g^{-1} \circ f$ $EVEN_{\mathbb{Z}^+} \to \mathbb{Z}^+$ is a bijection.

Note: To prove that a set A is denumerable, it is enough to find a bijection between A and another set we already know is denumerable (not necessarily $\mathbb{Z}^+$)

1. e $|A| = |B| \wedge |B| = |\mathbb{Z}^+| \Rightarrow |A| = |\mathbb{Z}^+|$

Th: denumerable sets are the "smallest" infinite sets: if A is an infinite set then there is an injective function $f \quad \mathbb{Z}^+ \longrightarrow A$

Proof: define $f(n)$ by recursion

$f(1) =$ pick one element $a_1 \in A$

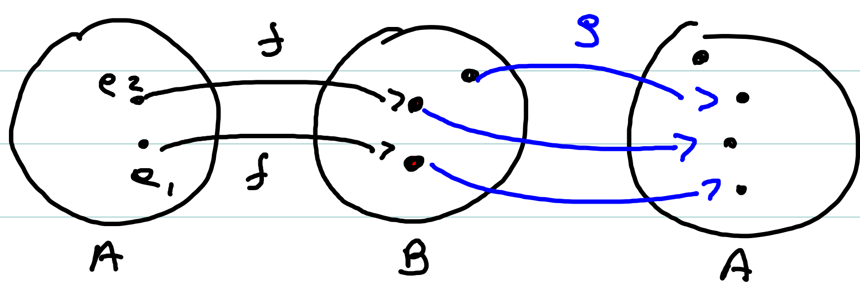
suppose $f(1)\ f(2) \cdots f(n)$ have been defined

$f(n+1) =$ pick one element $a_{n+1} \in A - \{f(1) \cdots f(n)\}$

Th: Cantor - Schröder Bernstein

Suppose there are injective functions $f\ A \to B$ and $g\ B \to A$ then there is a bijection $h\ A \to B$

Proof not given



i.e $|A| \leq |B| \wedge |B| \leq |A| \Rightarrow |A| = |B|$

Th $A = \{ n \in \mathbb{Z} / \ 3 \text{ divides } n \}$
is denumerable.

Proof :
idea 1 find a bijection $f: \mathbb{Z} \rightarrow A$

1) Define function $f \ \mathbb{Z} \rightarrow A$
$f(n) = 3n$

2) Prove f is a bijection (i.e prove f is injective and surjective or prove f has an inverse)

Idea 2. Since $A \subseteq \mathbb{Z}$

$f: A \to \mathbb{Z}$ is injective; since $\mathbb{Z}$
$f(a) = a$

is denumerable there is a bijection
$g: \mathbb{Z} \to \mathbb{Z}^+$

$g \circ f: A \to \mathbb{Z}^+$ is injective

Since A is infinite there is
an injective function $g: \mathbb{Z}^+ \to A$ (see previous
page) then by C-S-B th there
is a bijection between A and $\mathbb{Z}^+$

Note: this tells us that any infinite
subset of a denumerable set is denumerable.

i.e

A is infinite $\wedge |A| \leq |B| \wedge |B| = |\mathbb{Z}^+| \Rightarrow |A| = |\mathbb{Z}^+|$

To prove that a set A is denumereble you need to:

1) Choose a set B you already know is denumerable
2) Define a function $f\ A \to B$ or $f\ B \to A$
3) Prove f is a bijection

or

Show A is an infinite subset of a denumereble set.

Let $A = \{ S \subseteq \mathbb{Z}^+ \mid |S| = 1 \}$ is A denumerable?

yes

Proof: consider $f : \mathbb{Z}^+ \longrightarrow A$

$$f(x) = \{x\}$$

f is injective since if $x_1, x_2 \in \mathbb{Z}^+$ and $x_1 \neq x_2$ then $\{x_1\} \neq \{x_2\}$.

f is surjective since given $S \in A$ there is $x \in \mathbb{Z}^+$ s.t $S = \{x\}$ so $S = f(x)$

Th If A is denumerable and $x \notin A$
Then $A \cup \{x\}$ is denumerable.

Proof: assume A is denumerable
and $f \ \mathbb{Z}^+ \to A$ is a bijection

$$
\begin{tikzcd}
\mathbb{Z}^+ \arrow[r, "f"] \arrow[rr, bend left, "?"] & A & A \cup \{x\}
\end{tikzcd}
$$

$\mathbb{Z}^+$	A	$A \cup \{x\}$
1	a_1	x
2	a_2	a_1
3	a_3	a_2
4	a_4	a_3
5	a_5	a_4
$\vdots$	$\vdots$	$\vdots$

Note denumerable sets are those that can be listed: first element, second element, ...

define $g \quad \mathbb{Z}^+ \longrightarrow A \cup \{x\}$

$$g(1) = x$$
$$g(n) = f(n-1) \quad \text{if } n > 1$$

consider $h \quad A \cup \{x\} \longrightarrow \mathbb{Z}^+$

$$h(\alpha) = \begin{cases} 1 & \text{if } \alpha = x \\ f^{-1}(\alpha) + 1 & \text{if } \alpha \neq x \end{cases}$$

then $h(g(z)) = z$ for all $z \in \mathbb{Z}^+$

$g(h(\alpha)) = \alpha$ for all $\alpha \in A \cup \{x\}$

Therefore $h = g^{-1}$ and g is a bijection