Lesson 15

Cardinality

Recall

(1) The composition of two bijections is a bijection.

(2) f is a bijection $\Leftrightarrow$ f is invertible $\Leftrightarrow$ f^{-1} is invertible $\Leftrightarrow$ f^{-1} is a bijection.

$f \quad A \to B \qquad$ bijection

$f^{-1} \quad B \to A \qquad (f^{-1})^{-1} = f$

So f^{-1} is a bijection

Th: Let A be a set with n elements. Then $|P(A)| = 2^n$

Proof: order the elements of A any way you like: $a_1, a_2, \dots, a_n$

Let $B = \{ s \mid s \text{ is a binary string of length } n \}$

We know $|B| = 2^n$

Idea: find a bijection between B and $P(A)$, this tells us $P(A)$ and B have the same number of elements.

B	$P(A)$
s_1	$f(s_1)$
s_2	$f(s_2)$
s_{2^n}	$f(s_{2^n})$

Consider: $f \quad B \longrightarrow P(A)$

$$f(x_1 \cdots x_n) = \{a_i / \ x_i = 1\}$$

Example $A = \{a_1, a_2, a_3\}$

$f(101) = \{a_1, a_3\}$

f is a bijection because it is invertible. Consider

$g \quad P(A) \longrightarrow B$

$$g(S)_i = \begin{cases} 0 & \text{if } a_i \notin S \\ 1 & \text{if } a_i \in S \end{cases}$$

Example: $A = \{a_1, a_2, a_3\}$ $S = \{a_1, a_3\}$

$g(\{a_1, a_3\}) = 101$

then $f(g(S)) = f(x_1 \cdots x_i \cdots x_n)$

where $x_i = 1$ if $a_i \in S$ and $f(x_1 \cdots x_n) = S$
$x_i = 0$ if $a_i \notin S$

$g(f(x_1 \cdots x_n)) = g(S)$, where
$S = \{a_i / \ x_i = 1\}$ and $g(S) = x_1 \cdots x_n$

Therefore $g = f^{-1}$ and f is a bijection

Can we compare the size of infinite sets?
When do two sets have the same size?
Idea: when there is a bijection between them.

So $\mathbb{Z}^+$, $\mathbb{Z}$, have the same size.

$$\begin{array}{cc} -1 & 2 \\ -2 & 4 \\ -3 & 6 \\ \vdots & \vdots \end{array}$$

$$\begin{array}{cc} 0 & 1 \\ 1 & 3 \\ 3 & 5 \\ \vdots & \vdots \end{array}$$

Do $\mathbb{Z}^+$ and $ODD_{\mathbb{Z}^+}$ have the same size?

$\mathbb{Z}^+$	$ODD_{\mathbb{Z}^+}$
1	1
2	3
3	5
4	7
$\vdots$	$\vdots$

$$f : \mathbb{Z}^+ \longrightarrow ODD_{\mathbb{Z}^+}$$
$$f(z) = 2z - 1$$

is a bijection. (Prove this)

Are there infinite sets of different sizes? Yes $\mathbb{Z}^+$ and $\mathbb{R}$ do not have the same size.

What is a real number?

An integer followed by an infinite 0/9 string

$\sqrt{2} = 2.1415 \ldots\ldots\ldots\ldots$

$1 = 1.00 \ldots\ldots 0 \ldots\ldots$

$\frac{1}{2} = 0.50 \ldots\ldots 0 \ldots\ldots$

$\frac{1}{3} = 0.333 \ldots\ldots$

Cantor's diagonalization th.

Th : there is no bijection $f\ \mathbb{Z}^+ \to [0,1]$

Proof : by contradiction suppose $f\ \mathbb{Z}^+ \to [0,1]$ is a bijection

$f(1) = 0.\,x_{11}\,x_{12}\,x_{13}\ldots$

$f(2) = 0.\,x_{21}\,x_{22}\,x_{23}\ldots$

$f(3) = 0.\,x_{31}\,x_{32}\,x_{33}\ldots$

We want to find $y \in [0\ 1]$ $y \neq f(n)$ for all n

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \end{pmatrix}$$

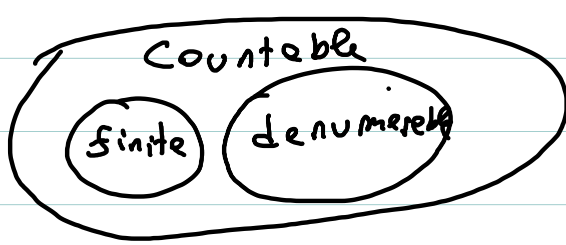
$y = 0.\,y_1\,y_2\,y_3\ldots$

$$y_i = \begin{cases} 2 & \text{if } x_{ii} = 3 \\ 3 & \text{if } x_{ii} \neq 3 \end{cases}$$

Def A and B have the same cardinality, or are equipotent, we write $|A| = |B|$, if there is a bijection $f\ A \to B$ (or $f\ B \to A$)

A has smaller cardinality then B, we write $|A| < |B|$, if there is $f\ A \to B$ injective but there is no bijection $f\ A \to B$

Def: sets that have the same cardinality as $\mathbb{Z}^+$ are called denumerable.
Sets there are either finite or denumerable are called countable



Now we know

$|z| = |z^+| = |OOD_{z^+}|$

$|z| < |[0\ 1]|$