Lesson 14

More about functions

Consider $f \quad \mathbb{Z}^+ \to ODD_{\mathbb{Z}^+} = \{x \in \mathbb{Z}^+, x \text{ is odd}\}$

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ 2n-1 & \text{if } n \text{ is odd} \end{cases}$$

Is f injective, is f surjective?

scratchwork:

$f(1) = 1 \qquad f(2) = 3$

$f(3) = 5 \qquad f(4) = 5$

$f(5) = 9 \qquad f(6) = 7$

$\vdots \qquad\qquad \vdots$

f is not injective because $f(3) = f(4)$

f is surjective: I need to prove

$\forall y \in ODD_{\mathbb{Z}^+} \ \exists x \in \mathbb{Z}^+ \quad y = f(x)$

If $y = 1$ take $x = 1$: $f(1) = 2 \cdot 1 - 1 = 1$

If $y > 1$ take $x = y - 1$ then

$x \in \mathbb{Z}^+$ and x is even so $f(x) = y - 1 + 1 = y$

Th : The composition of two injective functions is injective

((f injective $\wedge$ g injective) $\Rightarrow$ $g \circ f$ injective .

Assume $f: A \to B$ $g: B \to C$ and f and g are injective.

Goal : show $g \circ f$ is injective , that is prove $\forall x_1, x_2 \in A$ $g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$

Assume x_1, x_2 are in A and $g(f(x_1)) = g(f(x_2))$ then

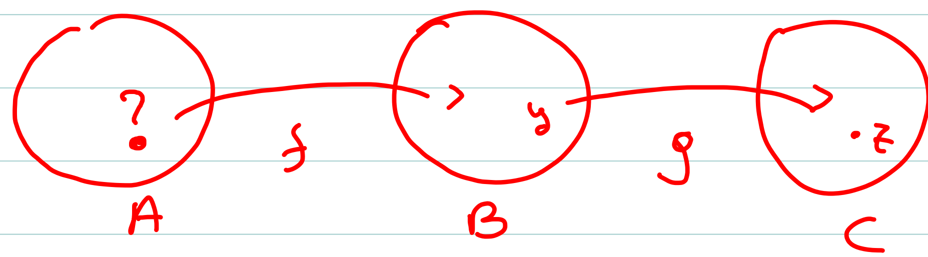
$f(x_1) = f(x_2)$, because g is injective

and $x_1 = x_2$ because f is injective, therefore $g \circ f$ is injective th: The composition of two surjective functions is surjective.

((f surjective $\wedge$ g surjective) $\Rightarrow$ $g \circ f$ surjective

Assume $f: A \to B$ $\quad g: B \to C$
and f and g are surjective

Goal : show $g \circ f$ is surjective, that is prove $\forall z \in C \ \exists x \in A \ g(f(x)) = z$



Given z in C there is y in B s.t $g(y) = z$ since g is surjective, and there is x in A s.t $f(x) = y$ since f is surjective. Therefore $g(f(x)) = g(y) = z$ so $g \circ f$ is surjective.

Th: the composition of two bijections is a bijection.

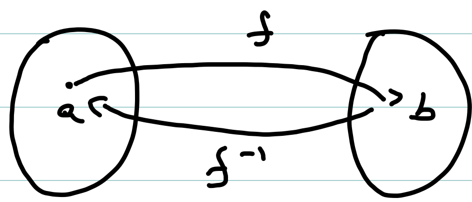
Def : Given f $A \to B$, the inverse of f, when it exists, is a function f^{-1} $B \to A$ s.t

$f^{-1}(b) = a \iff f(a) = b$

or equivalently satisfying

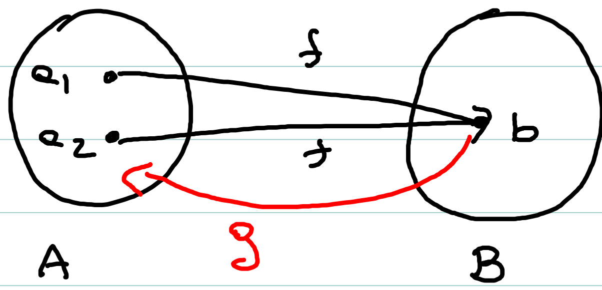
i) $f^{-1}(f(a)) = a$ for all $a \in A$

ii) $f(f^{-1}(b)) = b$ for all $b \in B$



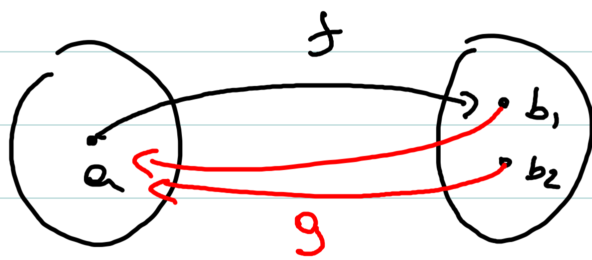
Question : can a function satisfy i) and not ii) or vice versa ?

If f^{-1} exists we say f is invertible.



$\forall\, b \in B \qquad f(g(b)) = b$

But $\qquad g(f(a_1)) \neq a_1$



$\forall\, a \in A \qquad g(f(a)) = a$

But

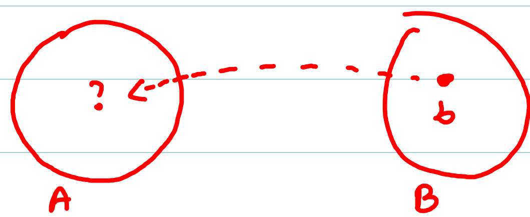
$f(g(b_2)) \neq b_2$

Th : f is invertible $\Leftrightarrow$ f is a bijection

Proof : first we shall prove f is a bijection $\Rightarrow$ f is invertible.

Assume $f \; A \to B$ is a bijection.

We want to define $g \; B \to A$ s.t $g = f^{-1}$



given $b \in B$ there is $a \in A$ s.t $f(a) = b$ because f is surjective, and a is unique because f is injective; define $g(b) = a$ (the unique element of A s.t $f(a) = b$) then clearly $g = f^{-1}$

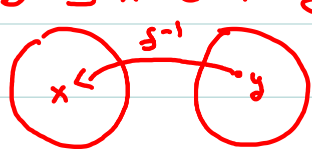
Now we need to prove : f is invertible $\Rightarrow f$ Injective
and f is invertible $\Rightarrow f$ surjective

Assume f is invertible.

To prove f is injective I need to prove
$\forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Assume $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$
then
$$f^{-1}(f(x_1)) = f^{-1}(f(x_2))$$
Therefore $x_1 = x_2$ so f is injective.

To prove f is surjective I need to prove
$\forall y \in B \; \exists x \in A \quad y = f(x)$



Suppose $y \in B$ let $x = f^{-1}(y)$ then $x \in A$
and $f(f^{-1}(y)) = y$ so f is surjective.

Th : $|P(\{1, 2, \dots n\})| = 2^n$

Let $A = \{1, 2, \dots n\}$

$B = \{ s \mid s \text{ is a binary string of length } n \}$

We know $|B| = 2^n$

Idea : find a bijection between P(A) and B, this tells us P(A) and B have the same number of elements.

Consider: $f \quad P(A) \longrightarrow B$

$$f(S)_i = \begin{cases} 0 \text{ if } i \notin S \\ 1 \text{ if } i \in S \end{cases}$$

Example: $A = \{1, 2, 3\}$
$f(\{1,3\}) = 101$

f is a bijection because it is invertible. Consider

$g \quad B \longrightarrow P(A)$

$$g(x_1 \cdots x_n) = \{i \mid x_i = 1\}$$

Example: $g(101) = \{1, 3\}$

then $g(f(S)) = g(x_1 \cdots x_n)$

where $x_i = 1$ if $i \in S$, $x_i = 0$ if $i \notin S$ and $g(x_1 \cdots x_n) = S$

$f(g(x_1 \cdots x_n)) = f(S)$, where $S = \{i \mid x_i = 1\}$ and $f(S) = x_1 \cdots x_n$

Therefore $g = f^{-1}$ and f is a bijection