Lesson 13

Injections, surjections, bijections

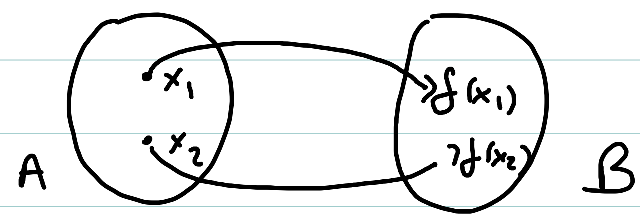
Def $f \; A \longrightarrow B$ is

a) Injective if $\forall x_1 \in A \; \forall x_2 \in A \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

b) surjective if $\forall y \in B \; \exists x \in A \quad y = f(x)$

c) Bijective if it is injective and surjective

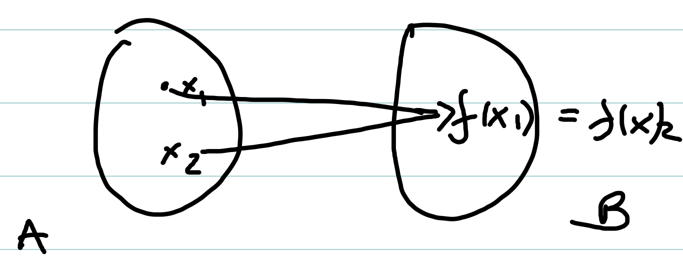
Injectivity:



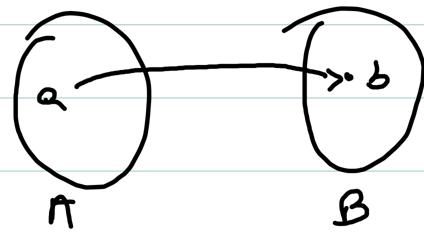
$\forall x_1 \in A \; \forall x_2 \in A \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
(contrapositive of statement in definition)

f not injective if

$\exists x_1 \in A \; \exists x_2 \in A \quad x_1 \neq x_2 \wedge f(x_1) = f(x_2)$

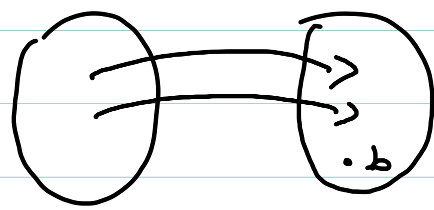


Surjectivity



f is not surjective if

$$\exists y \in B \ \forall x \in A \quad f(x) \neq y$$



Examples:

1) $f \quad \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$

$$f(x) = x^2$$

injective: we will prove
$\forall x_1, x_2 \in A \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

suppose $\sqrt{x_1^2} = \sqrt{x_2^2}$ and $x_1 > 0, x_2 > 0$
then $x_1 = \sqrt{x_1^2} = \sqrt{x_2^2} = x_2$ by algebre

Not surjective: we need to prove
$\exists y \in \mathbb{Z}^+ \ \forall x \in \mathbb{Z}^+ \quad y \neq f(x)$

Take $y = 2$ then $2 \neq x^2$ for all x in $\mathbb{Z}^+$

2) $g \quad \mathbb{Z} \longrightarrow \mathbb{Z}$

$g(x) = x^2$

not injective since $g(1) = g(-1)$

not surjective (same argument as for f)

3) $h \quad \mathbb{R}^+ \longrightarrow \mathbb{R}$

$h(x) = x^2$

Injective : suppose $x_1^2 = x_2^2$ and x_1, x_2 in $\mathbb{R}$ $x_1 > 0$, $x_2 > 0$ then

$x_1 = \sqrt{x_1^2} = \sqrt{x_1^2} = x_2$

Not surjective : take $-1 \in \mathbb{R}$ then $-1 \neq x^2$ for all x in $\mathbb{R}$

4) $j \quad \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$j(x) = x^2$

Injective (same argument as for f or h)

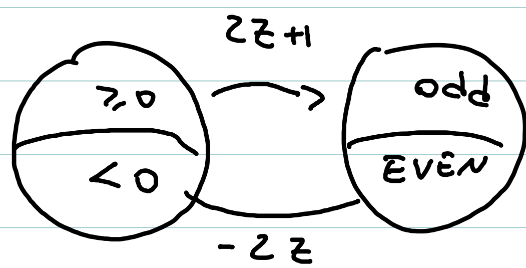
Surjective: given $y \in \mathbb{R}^+$ take $x = \sqrt{y}$ then $f(x) = (\sqrt{y})^2 = y$

$f \quad \mathbb{Z} \longrightarrow \mathbb{Z}^+$

$$f(z) = \begin{cases} 2z+1 & \text{if } z \geq 0 \\ -2z & \text{if } z < 0 \end{cases}$$

z	$f(z)$
0	1
1	3
2	5
3	7
4	9
⋮	⋮

z	$f(z)$
-1	2
-2	4
-3	6
-4	8
⋮	⋮



f is injective : assume $x_1 \neq x_2$ then

a) If x_1 x_2 both ≥ 0 $2x_1+1 \neq 2x_2+1$

b) If x_1 x_2 both < 0 $-2x_1 \neq -2x_2$

c) If one of x_1 x_2 ≥ 0 and the other < 0, wlog $x_1 \geq 0$ and $x_2 < 0$, then $f(x_1)$ is odd and $f(x_2)$ is even so certainly $f(x_1) \neq f(x_2)$

f is surjective : given $y \in Z^+$ if y is odd take $x = \frac{y-1}{2}$ then $x \geq 0$ and $x \in Z$ so

$f(x) = 2x+1 = 2\left(\frac{y-1}{2}\right)+1 = y$

if y is even, take $x = -\frac{y}{2}$ then

$x < 0$ and $x \in Z$ so $f(x) = -2x =$

$-2\left(-\frac{y}{2}\right) = y$

so $\forall y \in Z^+ \ \exists x \in Z \ \ y = f(x)$

Are there more integers in $\mathbb{Z}$ than $\mathbb{Z}^+$?