Lesson 12

Nested quantifiers

Functions

Recall

$\forall x$ in S $P(x)$

means all x in S have property P.

$\exists x$ in S $P(x)$

means there is x in S that has property P.

$\exists! x$ in S $P(x)$

means there is a unique x in S that has property P.

Negation rules:

$\neg\, \forall x \in A\ \exists y \in B \quad P(x) \Rightarrow (Q(y) \vee \exists z \in C \quad x \leq z)$

means

$\exists x \in A\ \forall y \in B \quad P(x) \wedge \neg Q(y) \wedge (\forall z \in C\ \ x > z)$

Nested quantifiers

$\forall m \in \mathbb{Z}^+ \ \exists n \in \mathbb{Z}^+ \quad m < n$ is True

n can depend on m

Proof: given m in $\mathbb{Z}^+$ take $\underbrace{n = m+1}$, then $m < n$

n depends on m
i.e every m is allowed to have its own n

$\underbrace{\exists n \in \mathbb{Z}^+} \ \forall m \in \mathbb{Z}^+ \quad m < n$ is False

n cannot depend on m

Proof: the negation of this statement is

$\forall n \in \mathbb{Z}^+ \ \exists m \in \mathbb{Z}^+ \quad m \geq n$:

given n in $\mathbb{Z}^+$, take $m = n$ then $m \geq n$.

Note: changing the order of different quantifiers $\forall \exists$ to $\exists \forall$ or vice versa does not produce, in general, an equivalent statement, so it is NOT OK

$\exists n \in \mathbb{Z}^+ \ \forall m \in \mathbb{Z}^+ \ m \geq n$ is true

Proof: Take $\underbrace{n = 1}$ then given $m \in \mathbb{Z}^+$ $m \geq 1$

n does not depend on m

$\exists n \in \mathbb{Z}^+ \ \forall m \in \mathbb{Z}^+ \ m > n$ is False

Proof: the negation of the statement is $\forall n \in \mathbb{Z}^+ \ \exists m \in \mathbb{Z}^+ \ m \leq n$. To prove this we argue: given n in $\mathbb{Z}^+$ take $m = n$ then $m \leq n$ is true.

Def: a function $f\ A \to B$ is defined by giving

1) The domain A, a nonempty set
2) The codomain B, a non empty set
3) A rule that to every element $x \in A$ associates an element $y = f(x) \in B$

input, preimage of y

output

image of x

Formally a rule is a set R:

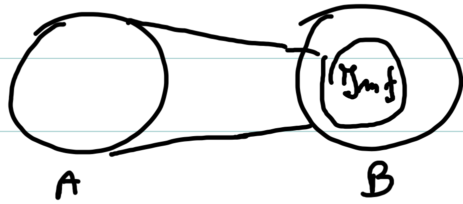
- $R \subseteq A \times B$
- $\forall x \in A\ \exists!\ z \in R$ the firs element of z is x

Note: $f\ \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ and $g\ \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$

are different functions, although they have the same rule

Def : The image or range of a function $f \; A \to B$ is the set of all outputs of f, that is the following subset of B:

$$Im(f) : \{ y \in B \mid \exists x \in A \quad y = f(x) \}$$
$f(A)$

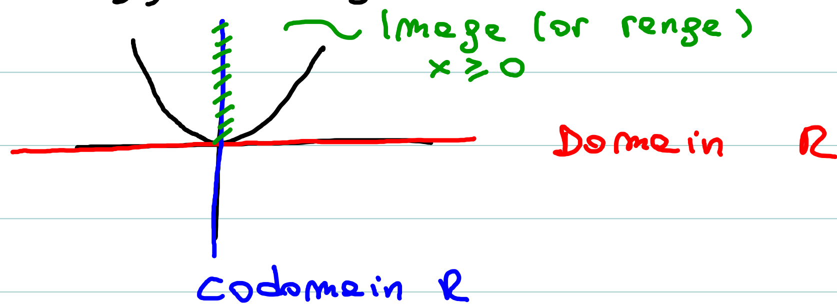


Ex $\quad f \quad R \to R$

$$f(x) = x^2$$

$$Im(f) = \{ x \in R \mid x \geq 0 \}$$

So in this case the image of f is different from the codomain R



Composition Given $f\ A \to B$ and $g\ B \to C$ then $g \circ f$, the composition of f and g is defined as follow

1) The domain of $g \circ f$ is A (the domain of f)
2) The codomain of $g \circ f$ is C (the codomain of g)
3) The rule is : $\forall x \in A \ (g \circ f)(x) = g(f(x))$

This is the usual definition you know from precalculus.

Note If $f : A \to A$
f^2 means $f \circ f$
f^3 means $f \circ f \circ f$
f^4 means $f \circ f \circ f \circ f \cdots$

This is the usual definition you know from precalculus.

Def : Given a non empty set A, the identy function on A, Id_A is defined by:

1) Domain is A
2) Codomain is A
3) Rule $\forall x \in A \quad Id_A(x) = x$

Def Given $f \; A \to B$ and a non empty subset S of A the restriction of f on S, $f_{|S}$, is defined by:

1) Domain S
2) Codomain B
3) Rule: $f_S(x) = f(x)$ for all x in S