Lesson 11

Cartesian product
Power set

Note :

(1, 2) is used in 2 different ways in math :

1) Ordered pair, i.e list of two elements
first element is 1, second element is 2
$(1,2) \neq (2,1)$

2) Interval on real line :
$(1,2) = \{x \in \mathbb{R} \mid 1 < x < 2\}$
$(2,1)$ not used

Def If A and B are non empty sets then

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

ordered pair

Ex $\{1,2\} \times \{3\} = \{(1,3)\ (2,3)\}$

$\{3\} \times \{1,2\} = \{(3,1)\ (3,2)\}$

Note: in general $A \times B \neq B \times A$

Q: can you find a necessary and sufficient condition for $A \times B = B \times A$?

Ex $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x\ y) \mid x \in \mathbb{R}, y \in \mathbb{R} \}$

Th $\forall A, B, C \quad \underbrace{A \times (B \cup C)}_{S_1} = \underbrace{(A \times B) \cup (A \times C)}_{S_2}$

Proof: first we shall prove $S_1 \subseteq S_2$:
assume $x \in S_1$ then $x = (a, y)$ with
$a \in A$ and $y \in B$ or $y \in C$. If $y \in B$
then $x \in A \times B$, if $y \in C$ then
$x \in A \times C$; in both cases $x \in (A \times B) \cup (A \times C)$

Now let's prove $S_2 \subseteq S_1$. Assume
$x \in S_2$; then $x \in A \times B$ or $x \in A \times C$
so $x = (a, y)$ and $a \in A$ and $y \in B$ or
$y \in C$ so $y \in B \cup C$, therefore
$x \in A \times (B \cup C)$

Def: if A is a set $P(A) = \{ S \mid S \subseteq A \}$

Ex $P(\phi) = \{ \phi \}$

$P(\{1\}) = \{ \phi, \{1\} \}$

$P(\{1,2\}) = \{ \phi, \{1\}, \{2\}, \{1,2\} \}$

Are the statements below T or F?

$1 \in \mathbb{Z}^+$ T

$1 \in P(\mathbb{Z}^+)$ F

$\{1,2\} \in \mathbb{Z}^+$ F

$\{1,2\} \in \mathbb{Z}^+ \times \mathbb{Z}^+$ F

$\{1,2\} \in P(\mathbb{Z}^+)$ T

$\{1,2\} \subseteq \mathbb{Z}^+$ T

$\{1,2\} \subseteq P(\mathbb{Z}^+)$ F

Th : If A has n elements P(A) has 2^n elements.

Proof by induction on n

We will prove $\forall n \geq 0 \ \underbrace{|A|}_{\text{\# elements of A}} = n \Rightarrow |P(A)| = 2^n$

1) Base case : if $n = 0$ $A = \emptyset$ and $P(A) = \{\emptyset\}$ so $|A| = 0$ and $|P(A)| = 1 = 2^0$

Inductive step : assume that $|P(A)| = 2^n$ whenever $|A| = n$. Suppose $|B| = n+1$ and $b \in B$ (B has at least one element) then

$$P(B) = \underbrace{\{S \mid S \subseteq B, \ b \in S\}}_{C} \cup \underbrace{\{S \mid S \subseteq B \ \ b \notin S\}}_{D}$$

1) $D = P(B - \{b\})$ so D has 2^n elements

2) $C = \{S \cup \{b\} \mid S \in D\}$ So $|C| = |D|$

So $|P(B)| = 2^n + 2^n = 2^{n+1}$

Alternative proof : let A
be a set with n elements
Every $S \subseteq A$ corresponds to a 0-1
string of length n and there are
2^n such strings.

Order the elements of A in some
arbitrary way, say $a_1, a_2 \dots a_n$
then define

$$f \quad P(A) \to B = \{ s \mid s \text{ is a } 01 \text{ string of length } n \}$$

$$f(S)_i = \begin{cases} 1 & \text{if } a_i \in S \\ 0 & \text{if } a_i \notin S \end{cases}$$

f is a bijection (i.e one to one
and onto), so $P(A)$ has as
many elements as B

For hw think about $P(P(A))$

$S \in P(P(A))$ if $S \subseteq P(A)$ so

$S = \{ X \mid X \subseteq A \}$ or $S = \emptyset$

Ex $P(\{1\}) = \{ \emptyset, \{1\} \}$

$P(P(\{1\})) = \{ \emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\} \}$

Th: $P(A \cap B) = P(A) \cap P(B)$

Proof: suppose $S \in P(A \cap B)$ then $S \subseteq A \cap B$ therefore $\forall x \in S$, $x \in A \cap B$ so $x \in A$ and $x \in B$. This means $S \subseteq A$ and $S \subseteq B$ so $S \in P(A)$ and $S \in P(B)$, therefore $S \in P(A) \cap P(B)$ so $P(A \cap B) \subseteq P(A) \cap P(B)$

Now assume $S \in P(A) \cap P(B)$ then $S \in P(A)$ and $S \in P(B)$ so $S \subseteq A$ and $S \subseteq B$ therefore $\forall x \in S$ $x \in A$ and $x \in B$ so $x \in A \cap B$; this means $S \subseteq A \cap B$ so $S \in P(A \cap B)$ so $P(A) \cap P(B) \subseteq P(A \cap B)$.

From both inclusions we derive $P(A \cap B) = P(A) \cap P(B)$

Do you think $P(A \cup B) = P(A) \cup P(B)$?

Russel's paradox

If a set can belong to itself
then we can look at:

$$A = \{ S \mid S \text{ is a set and } S \notin S \}$$

Question: does $A \in A$?