Math 300 A Spring 2023 Final

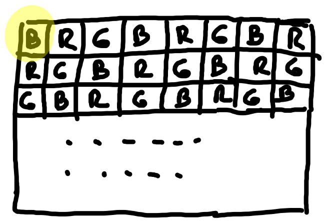
NAME (First,Last) : ....

Student ID ....

UW email ....

- Please write your name as it appears in the Canvas 's roster.
- IMPORTANT: Your exam will be scanned: DO NOT write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of the second page of this exam.
- Do not turn in any scratch paper.
- Unless stated otherwise, you **MUST** justify your answers and explain why your examples work.
- Your work needs to be neat and legible.

Problem 1 Prove that an 8x8 checkerboard with the square in position (1,1) , that is top row and leftmost column removed cannot be covered by 1x3 tiles.



Let's color the squares in the board R, B, G, starting with the top right corner, as the drawing shows. We will have a total of 22 R, 21 B, 21 G. After we remove the top left square we are left with 22 R, 20 B and 21 green squares. Every 1x3 tile placed on the board covers one R, one G, one B square, therefore since there are more R squares on the board, then B or G, the board cannot be covered by 1x3 tiles.

Problem 2 Let A be the subset of the interval [0, 1], containing all real numbers of the form $0.x_1x_2\cdots x_n\cdots$ where each decimal digit x_n is either 4 or 7. Is A denumerable ? Prove your answer.

No suppose $f\ \mathbb{Z}^+ \longrightarrow A$ consider the number $y = 0.y_1 y_2 \cdots y_n \cdots$ with

$$y_i = \begin{cases} 4 & \text{if } f(i)_i = 7 \\ 7 & \text{if } f(i)_i = 4 \end{cases}$$

then $y \in A$, and $y \neq f(i)$ for all $i \in \mathbb{Z}^+$ so f is not surjective.
This proves that no function $f\ \mathbb{Z}^+ \to A$ can be surjective, therefore there cannot be a bijection between A and $\mathbb{Z}^+$.

$\boxed{\text{A is not denumerable}}$

Problem 4 Prove that for all nonempty sets A,B and C,

$$(C \times C) - (A \times B) = ((C - A) \times C) \cup (C \times (C - B))$$

Let $S_1 = (C \times C) - (A \times B)$. $S_2 = (C-A) \times C \cup C \times (C-B)$

First we shall show $S_1 \subseteq S_2$:

Suppose $x \in S_1$ then x is a pair, $x = (y, z)$ and $(y, z) \in C \times C$ and $(y, z) \notin A \times B$, therefore $y, z \in C$ and $y \notin A$ or $z \notin B$. If $y \notin A$ then $(y, z) \in C-A \times C$; if $z \notin B$ then $(y, z) \in C \times C-B$

In any case $(y, z) \in S_2$

Now we shall show $S_2 \subseteq S_1$:

suppose $x \in S_2$, then $x \in ((C-A) \times C$ or $x \in C \times (C-B))$;

if $x \in (C-A) \times C$ then $x = (y, z)$ with $y \in C-A$, $z \in C$

so $(y, z) \in C \times C$ and $(y\ z) \notin A \times B$ so $(y\ z) \in S_1$

if $x \in C \times (C-B)$ then $x = (y, z)$ with $y \in C$, and $z \in C-B$ so $(y, z) \in C \times C$ and $(y\ z) \notin A \times B$

so $(y, z) \in S_2$

Problem 4 Find integers a, b, c and m such that $ac \equiv bc \bmod m$ but $a \not\equiv b \bmod m$.

$a = 3$ $b = 1$ $c = 2$

$m = 4$

Then

$3 \cdot 2 \equiv 1 \cdot 2 \mod 4$ but

$3 \not\equiv 1 \mod 4$

Problem 5 Consider the function $f : Z_{13} \to Z_{13}$ defined by $f(x) = x^2$. Is f injective? Prove your answer

No $f(1) = 1$, $12^2 = 144 = 13 \times 11 + 1$ So
$f(12) = 1$

Consider the function $f : Z_{23} \to Z_{23}$ defined by $f(x) = 2x + 3$. Is f surjective ? Prove your answer.

yes Given $y \in Z$ to find x s.t $f(x) = y$ we need to solve $2x + 3 \equiv_{23} y$ that is $2x \equiv_{23} y - 3$ which has solution since $(2, 23) = 1$ and 1 div $y-3$ no matter what y is.

Problem 6 Prove that $\sum_{i=0}^{n} r^{i} = \frac{1-r^{n+1}}{1-r}$

By induction.

Base case if $n=0$ $\sum_{i=0}^{0} r^i = 1$ and $\frac{1-r^{0+1}}{1-r} = 1$

Inductive step: assume $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$, then

$$\sum_{i=0}^{n+1} r^i = \sum_{i=0}^{n} r^i + r^{n+1} = \frac{1-r^{n+1}}{1-r} + r^{n+1} =$$

$$= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} = \frac{1-r^{n+1+1}}{1-r}$$