Math 546

Spring 2019

Final Exam

Due Thursday June 13 at 1:00 PM in my office (C-554) or my mailbox. No late papers will be accepted (except for dire emergencies and with prior arrangement). Please be sure to keep a copy of your test.

You may use the text and your class notes. You may not use any other source without clearing it with me; you may also not consult with anyone else or use the internet in any way.

You may quote any results from (the covered portion of) your text, from class, and from any assigned homework exercises. You may also quote any result which is "well-known". If you are unsure of what you need to prove, please contact me.

You will be graded on your ability to write correct, complete, and coherent proofs. Make sure your notation is clear and makes sense.

As always, please write legibly. If you typeset your exam, please proofread carefully. It can be very difficult for me to distinguish a typographical error from a misunderstanding.

All manifolds are assumed to be smooth and without boundary unless indicated otherwise.

1. Compute the volume of S^3 (with the usual metric) using the Riemannian volume form of Problem 16-9.

2. Let G be a Lie group with unit e. Prove that there exists a neighborhood U of e such that, for each $x \in U$, there exists a unique $z \in U$ such that $z^2 = x$.

3. Let D_1 and D_2 be involutive distributions on the n-manifold M of dimensions d and $n-d$ respectively and suppose that $T_pM = (D_1)_p \oplus (D_2)_p$ for all $p \in M$. Prove that, for each $p \in M$, there exists a smooth chart (U, ϕ), where U is a neighborhood of p in M, such that D_1 is spanned by $\partial/\partial x^1, \ldots, \partial/\partial x^d$ and D_2 is spanned by $\partial/\partial x^{d+1}, \ldots, \partial/\partial x^n$ on U.

4. Suppose that M is a compact manifold and that $f : M \to N$ is smooth. If q is a regular value of f, prove that there exists a neighborhood U of q and a diffeomorphism $h : f^{-1}\{q\} \times U \longrightarrow f^{-1}(U)$ such that $f(h(x,y)) = y$. Note that this problem implies in particular that $f^{-1}\{q\}$ is diffeomorphic to $f^{-1}\{y\}$ for all y in a neighborhood of q. (Hint: You might want to start by trying to prove this for the case $N = \mathbb{R}$. Here you might want to construct a smooth vector field on M which is f-related to $\frac{d}{dt}$ in a neighborhood of $f^{-1}\{q\}$ and consider its flow. The general case can be proven by a similar strategy.)