Math 546

Spring 2019

Homework 6

Read the sections Line Integrals and Conservative Covector Fields in Chapter 11. (You might wish to postpone reading the section Conservative Covector Fields until after we discuss de Rham cohomology.) Then read Chapter 16, skipping the section Manifolds with Corners if you wish. Finally, I will talk a little about de Rham cohomology, roughly the first two sections of Chapter 17. Then do the following Problems.

From your text: 16-2, 16-4, 16-5, 16-6, and 16-9. We have not discussed how continuous maps can be replaced by smooth maps — this is in Chapter 6 — so you should assume in 16-4 that the retraction is smooth and in 16-5 that F and G are smoothly homotopic. (You may use what we prove about de Rham cohomology; this will make the hint unnecessary.) Also, replace (continuous) vector fields by smooth vector fields and homotopies by smooth homotopies in 16-6.

Supplementary exercises:

1. Suppose that $f : M \to N$ is a smooth map between compact connected orientable n-manifolds which is not surjective. Prove that if ω is any n-form on N, then $f^*\omega$ is exact. (Hint: You may use the fact, which we will probably not prove, that if N is as above and ω is an n-form with $\int_N \omega = 0$, then ω is exact.) This, along with the hint, is essentially an old prelim problem.

2. Let M be a smooth manifold, $I = [0, 1]$, and let $\pi : M \times I \to M$ denote the projection onto the first coordinate. Prove that any $\omega \in \Omega^k(M \times I)$, $k \geq 1$, can be written uniquely as

$$\omega = \omega' + dt \wedge \eta,$$

where ω' and η are smooth forms with

$$\omega'_{(p,s)}(v_1, \ldots, v_k) = 0$$

whenever any $v_i \in \ker d\pi_{(p,s)}$ and with η having the same property. Here $t : M \times I \to I$ is the projection onto the second coordinate.