Math 442

Winter 2019

Midterm Exam

NAME Solutions

Point totals are indicated in parentheses.

(10) 1. Prove that $\{ (x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } x^2 + y^2 \leq 1 \}$ is not a regular surface.

See solutions to HW 3.

(10) 2. Let $a \in \mathbb{R}$, $a \neq 0$, and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(s,t) = (sa, st, (1-s)t)\,.$$

Prove that $f(\mathbb{R}^2)$ is a regular surface and that $f : \mathbb{R}^2 \to f(\mathbb{R}^2)$ is a parametrization. (Hint: Prove that f is a homeomorphism onto its image by finding f^{-1}.)

If $(sa, st, (1-s)t) = (x, y, z)$, then $s = \frac{x}{a}$ and $t = y + z$.
Thus $f^{-1} : f(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ is given by $f^{-1}(x,y,z) = (\frac{x}{a}, y+z)$.
Since f^{-1} is continuous, $f : \mathbb{R}^2 \to f(\mathbb{R}^2)$ is a homeomorphism.

f is clearly smooth and

$$(df)(s,t) = \begin{bmatrix} a & 0 \\ t & s \\ -t & 1-s \end{bmatrix}.$$

This matrix has rank 2 since $\begin{vmatrix} a & 0 \\ t & s \end{vmatrix} = sa \neq 0$ if $s \neq 0$; and if $s = 0$, $\begin{vmatrix} a & 0 \\ -t & 1-s \end{vmatrix} = (1-s)a \neq 0$.

Therefore, $f(\mathbb{R}^2)$ is a regular surface and f is a parametrization.

(10) 3 Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a smooth curve parametrized by arc length with no singular points of order 1 and no points s for which the torsion $\tau(s) = 0$. Suppose that $\boldsymbol{\alpha}(I)$ is contained in a sphere with center the origin.

(2) a. Prove that $\boldsymbol{\alpha}(s) \cdot \boldsymbol{\alpha}'(s) = 0$ for all s.

(6) b. Prove that $\boldsymbol{\alpha}(s) = -R(s)\mathbf{n}(s) + R'(s)T(s)\mathbf{b}(s)$, where $R(s) = 1/k(s)$ and $T(s) = 1/\tau(s)$. (As usual, $\mathbf{n}(s)$ denotes the normal vector at s and $\mathbf{b}(s)$ denotes the binormal vector at s.)

(2) c. Conclude that $R(s)^2 + R'(s)^2T(s)^2$ is constant.

a. $\alpha(s)\cdot\alpha(s)$ is constant, so $0 = \frac{d}{ds}\left(\alpha(s)\cdot\alpha(s)\right) = 2\alpha(s)\cdot\alpha'(s)$

b. Since $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ is an orthonormal basis for each s, we have $\alpha(s) = c_1(s)\vec{t}(s) + c_2(s)\vec{n}(s) + c_3(s)\vec{b}(s)$, where $c_1(s) = \alpha(s)\cdot\vec{t}(s)$, $c_2(s) = \alpha(s)\cdot\vec{n}(s)$, and $c_3(s) = \alpha(s)\cdot\vec{b}(s)$.

By part a), $c_1(s) = 0$. Next, $c_2(s) = \frac{1}{k(s)}\alpha(s)\cdot\alpha''(s)$.

But by part a),

$$0 = \frac{d}{ds}\left(\alpha(s)\cdot\alpha'(s)\right) = \alpha'(s)\cdot\alpha'(s) + \alpha(s)\cdot\alpha''(s) = 1 + \alpha(s)\cdot\alpha''(s)$$

so $c_2(s) = -R(s)$. Finally,

$$\alpha(s)\cdot\vec{n}(s) = -R(s) \Rightarrow \frac{d}{ds}\left(\alpha(s)\cdot\vec{n}(s)\right) = -R'(s).$$

But

$$\begin{aligned}\frac{d}{ds}\left(\alpha(s)\cdot\vec{n}(s)\right) &= \alpha'(s)\cdot\vec{n}(s) + \alpha(s)\cdot\vec{n}'(s) \\ &= \alpha(s)\cdot\vec{n}'(s) \\ &= \alpha(s)\cdot\left(-\tau(s)\vec{b}(s) - k(s)\vec{t}(s)\right) \quad \text{(Frenet formula)} \\ &= -\tau(s)\,\alpha(s)\cdot\vec{b}(s) \quad \text{by part a).}\end{aligned}$$

This implies that $c_3(s) = R'(s)T(s)$.

c. By part b),

$$\alpha(s)\cdot\alpha(s) = \left[-R(s)\vec{n}(s) + R'(s)T(s)\vec{b}(s)\right]\cdot\left[-R(s)\vec{n}(s) + R'(s)T(s)\vec{b}(s)\right]$$

$$= R(s)^2 + R'(s)^2\,T(s)^2$$

since $\vec{n}(s)$ and $\vec{b}(s)$ are orthonormal. But $\alpha(s)\cdot\alpha(s)$ is constant.