2-5.3. Let \( x : \mathbb{R}^2 \to S^2 \setminus \{ N \} \) be given by

\[
x(u, v) = \left( \frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right),
\]

where here \( S^2 \) is the sphere of radius 1 with center at \((0, 0, 1)\) and \( N = (0, 0, 2) \). (Of course, \( x \) is just \( \pi^{-1} \) of exercise 16 in 2-2.) Now

\[
x_u(u, v) = \left( \frac{4(v^2 - u^2 + 4)}{(u^2 + v^2 + 4)^2}, -\frac{8uv}{(u^2 + v^2 + 4)^2}, \frac{16u}{(u^2 + v^2 + 4)^2} \right)
\]

\[
x_v(u, v) = \left( \frac{-8uv}{(u^2 + v^2 + 4)^2}, \frac{4(v^2 - u^2 + 4)}{(u^2 + v^2 + 4)^2}, \frac{16v}{(u^2 + v^2 + 4)^2} \right).
\]

If \((u_0, v_0) \in \mathbb{R}^2\),

\[
F(u_0, v_0) = \langle x_u, x_v \rangle_{x(u_0, v_0)} = 0
\]

\[
E(u_0, v_0) = \langle x_u, x_u \rangle_{x(u_0, v_0)} = \frac{16}{(u_0^2 + v_0^2 + 4)^2}
\]

\[
G(u_0, v_0) = \langle x_v, x_v \rangle_{x(u_0, v_0)} = \frac{16}{(u_0^2 + v_0^2 + 4)^2}.
\]

Therefore, if \( p = x(u_0, v_0) \), we have

\[
I_p(ax_u + bx_v) = \frac{16}{(u_0^2 + v_0^2 + 4)^2}(a^2 + b^2).
\]

2-5.11a. Parametrize \( C \) by arc length; as in Example 4 on page 78 of do Carmo, we let

\[
x = f(v), \quad z = g(v), \quad 0 < v < l, \quad f(v) > 0
\]

be this parametrization. We then have the parametrization \( x : U \to S \setminus C \) given by

\[
x(u, v) = (f(v) \cos u, f(v) \sin u, g(v)),
\]

where \( U = \{ (u, v) \in \mathbb{R}^2 : 0 < u < 2\pi, 0 < v < l \} \). Then the area of \( S \) is given by

\[
A(S) = \int_0^l \int_0^{2\pi} |x_u \wedge x_v| \, du \, dv.
\]

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(Actually this expression is not quite justified (why?); however, you need not provide a proof.)

Now
\[
x_u = (-f(v) \sin u, f(v) \cos u, 0)
\]
\[
x_v = (f'(v) \cos u, f'(v) \sin u, g'(v)),
\]
so
\[
|x_u \wedge x_v| = |(g'(v)f(v) \cos u, g'(v)f(v) \sin u, -f(v)f'(v))|
\]
\[
= f(v) \sqrt{g'(v)^2 + f'(v)^2} = f(v) = \rho(v).
\]

Hence,
\[
A(S) = \int_0^l \int_0^{2\pi} \rho(v) \, du \, dv = 2\pi \int_0^l \rho(v) \, dv.
\]

b. Let \( C \) be the circle in the \( xz \)-plane with radius \( r \) and center \((a, 0, 0)\), with \( r < a \). The surface obtained by rotating \( C \) around the \( z \)-axis is a torus \( T \). If we parametrize \( C \) by starting at \( x = a+r \) and proceeding counterclockwise, we obtain
\[
\rho(s) = a + r \cos \left(\frac{s}{r}\right) \quad 0 \leq s \leq 2\pi r.
\]

Then, by part a),
\[
A(T) = 2\pi \int_0^{2\pi r} a + r \cos \left(\frac{s}{r}\right) \, ds = 4\pi^2 ra.
\]

2-5.14 a. We have
\[
\langle \text{grad} \, f(p), x_u \rangle = df_p(x_u) = \frac{\partial(f \circ x)}{\partial u} = f_u
\]
\[
\langle \text{grad} \, f(p), x_v \rangle = df_p(x_v) = \frac{\partial(f \circ x)}{\partial v} = f_v.
\]

Write \( \text{grad} \, f(p) = ax_u + bx_v, \, a, b \in \mathbb{R} \). (Of course, \( a \) and \( b \) depend upon \( p \).) Then the above formulas yield
\[
aE + bF = f_u
\]
\[
aF + bG = f_v.
\]

Solve for \( a \) and \( b \) to get
\[
a = \frac{f_u G - f_v F}{EG - F^2},
\]
\[
b = \frac{f_u F - f_v E}{F^2 - EG}.
\]

Note that since \( x_u \) and \( x_v \) are linearly independent, \( x_u \wedge x_v \neq 0 \). Thus \( EG - F^2 = |x_u \wedge x_v|^2 > 0 \).
b. By the Cauchy-Schwarz inequality,
\[ df_p(v) \leq |(\text{grad } f(p), v)_p| \leq |\text{grad } f(p)| \]
whenever |v| = 1. If \( v = \text{grad } f(p)/|\text{grad } f(p)| \), this maximum is attained. Since equality holds in the Cauchy-Schwarz inequality only when \( v \) and \( \text{grad } f(p) \) are collinear, the maximum of \( df_p(v) \), \(|v| = 1\), occurs only when \( v = \text{grad } f(p)/|\text{grad } f(p)| \).

c. Let \( c \in \mathbb{R} \), and let \( C = f^{-1}(c) \subset S \). If \( \text{grad } f(p) \neq 0 \) for any \( p \in C \), then \( df_p : T_pS \to \mathbb{R} \) is onto and thus \( c \) is a regular value. It follows from Exercise 28 of 2-4 that \( C \) is a regular curve.

Suppose now that \( \alpha : I \to C \) is a smooth parametrized curve with \( \alpha(0) = p \). Then \( (f \circ \alpha)(t) = c \) for all \( t \in I \); this implies that
\[ 0 = (f \circ \alpha)'(0) = df_p(\alpha'(0)) = (\text{grad } f(p), \alpha'(0))_p. \]
Since any vector in \( T_pC \) is of the form \( \alpha'(0) \) for some such \( \alpha \), it follows that \( \text{grad } f(p) \) is normal to \( C \) at \( p \).

2-6.1. We prove the contrapositive. Suppose \( S \) is an orientable regular surface covered by connected coordinate neighborhoods \( V_1 \) and \( V_2 \). Choose an orientation on \( S \), and let \( x_1 : U_1 \to V_1 \) and \( x_2 : U_2 \to V_2 \) be parametrizations. Since \( U_i \) is connected, it follows that \( dx_i(q) : \mathbb{R}^2 \to T_{x_i(q)}S \) is either orientation preserving for all \( q \in U_i \) or orientation reversing for all \( q \in U_i \). Write \( \tau(x_i) = 1 \) if \( dx_i(q) \) is orientation preserving and \( \tau(x_i) = -1 \) if \( dx_i(q) \) is orientation reversing. Then the Jacobian determinant of \( x_2^{-1} \circ x_1 \) is always positive if \( \tau(x_1)\tau(x_2) = 1 \) and always negative if \( \tau(x_1)\tau(x_2) = -1 \). In particular, the sign of the Jacobian of \( x_2^{-1} \circ x_1 \) must be unchanging on \( V_1 \cap V_2 \).

2-6.2. We have that \( d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2 \) is an isomorphism since \( \varphi \) is a local diffeomorphism at \( p \). Choose the orientation on \( T_pS_1 \) such that \( d\varphi_p \) is orientation preserving. (We assume that we have chosen an orientation on \( S_2 \).) Now choose a connected coordinate neighborhood \( V \) of \( p \) such that \( \varphi|V : V \to \varphi(V) \) is a diffeomorphism onto an open set in \( S_2 \). Let \( x : U \to V \) be a parametrization. Then \( \varphi \circ x : U \to \varphi(V) \) is a parametrization and therefore \( d(\varphi \circ x)(q) \) is orientation preserving for all \( q \in U \) or orientation reversing for all \( q \in U \). Since \( d(\varphi \circ x)(q) = d\varphi(x(q)) \circ dx(q) \), and we have defined our orientation on each tangent plane of \( S_1 \) so that \( d\varphi \) is orientation preserving, it follow that \( dx(q) \) is either orientation preserving for all \( q \in U \) or orientation reversing for all \( q \in U \). Since \( p \in S_1 \) is arbitrary, this shows that \( S_1 \) is orientable.

2-6.4. Let \( S \) be a connected orientable regular surface, and let \( N : S \to \mathbb{R}^3 \) be an orientation. If \( M : S \to \mathbb{R}^3 \) is another orientation, we have that \( M(p) = \pm N(p) \) for each \( p \in S \). Consider the set \( C = \{ p \in S : M(p) = N(p) \} \). I claim that \( C \) is both open and closed in \( S \). Assuming this, it follows by the connectivity of \( S \) that either \( C = \emptyset \) or \( C = S \). If \( C = S \), we have that \( M = N \); if \( C = \emptyset \), we have that \( M = -N \). Therefore, \( S \) has exactly two orientations.

We now prove the claim. Since \( M \) and \( N \) are continuous, it is immediate that \( C \) is closed. But
\[ S \setminus C = \{ p \in S : M(p) = -N(p) \}, \]
and by the same argument, \( S \setminus C \) is closed. Therefore, \( C \) is open as well, completing the proof.