**Math 442**

**Winter 2019**

**Solutions to Homework 6**

2-5.3. Let $\mathbf{x}:\mathbb{R}^2\to S^2\setminus\{N\}$ be given by

$$\mathbf{x}(u,v)=\left(\frac{4u}{u^2+v^2+4},\frac{4v}{u^2+v^2+4},\frac{2(u^2+v^2)}{u^2+v^2+4}\right),$$

where here $S^2$ is the sphere of radius 1 with center at $(0,0,1)$ and $N=(0,0,2)$. (Of course, $\mathbf{x}$ is just $\pi^{-1}$ of exercise 16 in 2-2.) Now

$$\mathbf{x}_u(u,v)=\left(\frac{4(v^2-u^2+4)}{(u^2+v^2+4)^2},\frac{-8vu}{(u^2+v^2+4)^2},\frac{16u}{(u^2+v^2+4)^2}\right)$$
$$\mathbf{x}_v(u,v)=\left(\frac{-8uv}{(u^2+v^2+4)^2},\frac{4(u^2-v^2+4)}{(u^2+v^2+4)^2},\frac{16v}{(u^2+v^2+4)^2}\right).$$

If $(u_0,v_0)\in\mathbb{R}^2$,

$$F(u_0,v_0)=\langle\mathbf{x}_u,\mathbf{x}_v\rangle_{\mathbf{x}(u_0,v_0)}=0$$
$$E(u_0,v_0)=\langle\mathbf{x}_u,\mathbf{x}_u\rangle_{\mathbf{x}(u_0,v_0)}=\frac{16}{(u_0^2+v_0^2+4)^2}$$
$$G(u_0,v_0)=\langle\mathbf{x}_v,\mathbf{x}_v\rangle_{\mathbf{x}(u_0,v_0)}=\frac{16}{(u_0^2+v_0^2+4)^2}.$$

Therefore, if $p=\mathbf{x}(u_0,v_0)$, we have

$$I_p(a\mathbf{x}_u+b\mathbf{x}_v)=\frac{16}{(u_0^2+v_0^2+4)^2}(a^2+b^2).$$

2-5.11a. Parametrize $C$ by arc length; as in Example 4 on page 78 of do Carmo, we let

$$x=f(v),\qquad z=g(v),\qquad 0<v<l,\qquad f(v)>0$$

be this parametrization. We then have the parametrization $\mathbf{x}:U\to S\setminus C$ given by

$$\mathbf{x}(u,v)=(f(v)\cos u,f(v)\sin u,g(v)),$$

where $U=\{\,(u,v)\in\mathbb{R}^2:0<u<2\pi,0<v<l\,\}$. Then the area of $S$ is given by

$$A(S)=\int_0^l\int_0^{2\pi}|\mathbf{x}_u\wedge\mathbf{x}_v|\,du\,dv.$$

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(Actually this expression is not quite justified (why?); however, you need not provide a proof.)

Now

$$\mathbf{x}_u = (-f(v)\sin u, f(v)\cos u, 0)$$
$$\mathbf{x}_v = (f'(v)\cos u, f'(v)\sin u, g'(v)),$$

so

$$\begin{aligned}|\mathbf{x}_u \wedge \mathbf{x}_v| &= |(g'(v)f(v)\cos u, g'(v)f(v)\sin u, -f(v)f'(v))| \\ &= f(v)\sqrt{g'(v)^2 + f'(v)^2} = f(v) = \rho(v).\end{aligned}$$

Hence,

$$A(S) = \int_0^l \int_0^{2\pi} \rho(v)\, du\, dv = 2\pi \int_0^l \rho(v)\, dv.$$

b. Let $C$ be the circle in the $xz$-plane with radius $r$ and center $(a,0,0)$, with $r < a$. The surface obtained by rotating $C$ around the $z$-axis is a torus $T$. If we parametrize $C$ by starting at $x = a+r$ and proceeding counterclockwise, we obtain

$$\rho(s) = a + r\cos(\frac{s}{r}) \qquad 0 \le s \le 2\pi r.$$

Then, by part a),

$$A(T) = 2\pi \int_0^{2\pi r} a + r\cos(\frac{s}{r})\, ds = 4\pi^2 ra.$$

2-5.14 a. We have

$$\langle \operatorname{grad} f(p), \mathbf{x}_u\rangle = df_p(\mathbf{x}_u) = \frac{\partial(f\circ\mathbf{x})}{\partial u} \equiv f_u$$
$$\langle \operatorname{grad} f(p), \mathbf{x}_v\rangle = df_p(\mathbf{x}_v) = \frac{\partial(f\circ\mathbf{x})}{\partial v} \equiv f_v.$$

Write $\operatorname{grad} f(p) = a\mathbf{x}_u + b\mathbf{x}_v$, $a, b \in \mathbb{R}$. (Of course, $a$ and $b$ depend upon $p$.) Then the above formulas yield

$$aE + bF = f_u$$
$$aF + bG = f_v.$$

Solve for $a$ and $b$ to get

$$a = \frac{f_u G - f_v F}{EG - F^2}$$
$$b = \frac{f_u F - f_v E}{F^2 - EG}.$$

Note that since $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent, $\mathbf{x}_u \wedge \mathbf{x}_v \neq 0$. Thus $EG - F^2 = |\mathbf{x}_u \wedge \mathbf{x}_v|^2 > 0$.

b. By the Cauchy-Schwarz inequality,

$$df_p(v) \leq |\langle \operatorname{grad} f(p), v\rangle_p| \leq |\operatorname{grad} f(p)|$$

whenever $|v| = 1$. If $v = \operatorname{grad} f(p)/|\operatorname{grad} f(p)|$, this maximum is attained. Since equality holds in the Cauchy-Schwarz inequality only when $v$ and $\operatorname{grad} f(p)$ are collinear, the maximum of $df_p(v)$, $|v| = 1$, occurs only when $v = \operatorname{grad} f(p)/|\operatorname{grad} f(p)|$.

c. Let $c \in \mathbb{R}$, and let $C = f^{-1}(c) \subset S$. If $\operatorname{grad} f(p) \neq 0$ for any $p \in C$, then $df_p : T_pS \to \mathbb{R}$ is onto and thus $c$ is a regular value. It follows from Exercise 28 of 2-4 that $C$ is a regular curve.

Suppose now that $\alpha : I \to C$ is a smooth parametrized curve with $\alpha(0) = p$. Then $(f \circ \alpha)(t) = c$ for all $t \in I$; this implies that

$$0 = (f \circ \alpha)'(0) = df_p(\alpha'(0)) = \langle \operatorname{grad} f(p), \alpha'(0)\rangle_p.$$

Since any vector in $T_pC$ is of the form $\alpha'(0)$ for some such $\alpha$, it follows that $\operatorname{grad} f(p)$ is normal to $C$ at $p$.

2-6.1. We prove the contrapositive. Suppose $S$ is an orientable regular surface covered by connected coordinate neighborhoods $V_1$ and $V_2$. Choose an orientation on $S$, and let $\mathbf{x}_1 : U_1 \to V_1$ and $\mathbf{x}_2 : U_2 \to V_2$ be parametrizations. Since $U_i$ is connected, it follows that $d\mathbf{x}_i(q) : \mathbb{R}^2 \to T_{\mathbf{x}_i(q)}S$ is either orientation preserving for all $q \in U_i$ or orientation reversing for all $q \in U_i$. Write $\tau(\mathbf{x}_i) = 1$ if $d\mathbf{x}_i(q)$ is orientation preserving and $\tau(\mathbf{x}_i) = -1$ if $d\mathbf{x}_i(q)$ is orientation reversing. Then the Jacobian determinant of $\mathbf{x}_2^{-1} \circ \mathbf{x}_1$ is always positive if $\tau(\mathbf{x}_1)\tau(\mathbf{x}_2) = 1$ and always negative if $\tau(\mathbf{x}_1)\tau(\mathbf{x}_2) = -1$. In particular, the sign of the Jacobian of $\mathbf{x}_2^{-1} \circ \mathbf{x}_1$ must be unchanging on $V_1 \cap V_2$.

2-6.2. We have that $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ is an isomorphism since $\varphi$ is a local diffeomorphism at $p$. Choose the orientation on $T_pS_1$ such that $d\varphi_p$ is orientation preserving. (We assume that we have chosen an orientation on $S_2$.) Now choose a connected coordinate neighborhood $V$ of $p$ such that $\varphi|V : V \to \varphi(V)$ is a diffeomorphism onto an open set in $S_2$. Let $\mathbf{x} : U \to V$ be a parametrization. Then $\varphi \circ \mathbf{x} : U \to \varphi(V)$ is a parametrization and therefore $d(\varphi \circ \mathbf{x})(q)$ is orientation preserving for all $q \in U$ or orientation reversing for all $q \in U$. Since $d(\varphi \circ \mathbf{x})(q) = d\varphi(\mathbf{x}(q) \circ d\mathbf{x}(q)$, and we have defined our orientation on each tangent plane of $S_1$ so that $d\varphi$ is orientation preserving, it follow that $d\mathbf{x}(q)$ is either orientation preserving for all $q \in U$ or orientation reversing for all $q \in U$. Since $p \in S_1$ is arbitrary, this shows that $S_1$ is orientable.

2-6.4. Let $S$ be a connected orientable regular surface, and let $N : S \to \mathbb{R}^3$ be an orientation. If $M : S \to \mathbb{R}^3$ is another orientation, we have that $M(p) = \pm N(p)$ for each $p \in S$. Consider the set $C = \{\, p \in S : M(p) = N(p) \,\}$. I claim that $C$ is both open and closed in $S$. Assuming this, it follows by the connectivity of $S$ that either $C = \emptyset$ or $C = S$. If $C = S$, we have that $M = N$; if $C = \emptyset$, we have that $M = -N$. Therefore, $S$ has exactly two orientations.

We now prove the claim. Since $M$ and $N$ are continuous, it is immediate that $C$ is closed. But

$$S \setminus C = \{\, p \in S : M(p) = -N(p) \,\},$$

and by the same argument, $S \setminus C$ is closed. Therefore, $C$ is open as well, completing the proof.