2-4.2. Let \( S = \{ (x, y, z) : x^2 + y^2 - z^2 = 1 \} \), and suppose \( \alpha : I \to S \) is a curve with \( \alpha(0) = (x, y, 0) \). Write \( \alpha(t) = (x(t), y(t), z(t)) \). Since \( x(t)^2 + y(t)^2 - z(t)^2 = 1 \) for all \( t \), we may differentiate both sides at \( t = 0 \) to obtain

\[
2x(0)x'(0) + 2y(0)y'(0) - 2z(0)z'(0) = 0.
\]

But \( (x(0), y(0), z(0)) = (x, y, 0) \); therefore \( \alpha'(0) \) is orthogonal to \( (x, y, 0) \). Since the set of vectors orthogonal to \( (x, y, 0) \) is a plane, this must be \( T_{(x,y,0)}S \), and the tangent plane is parallel to the \( z \)-axis.

2-4.3. Let \( S \) be the graph of the differentiable function \( z = f(x, y) \). Then \( x : U \to S \) defined by \( x(u, v) = (u, v, f(u, v)) \) is a parametrization. Let \( (x_0, y_0) \in U \) and \( z_0 = f(x_0, y_0) \). The tangent space to \( S \) at \( (x_0, y_0, z_0) \) is given by \( (dx)(x_0, y_0)(\mathbb{R}^2) \). But

\[
(dx)(x_0, y_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x(x_0, y_0) & f_y(x_0, y_0) \end{bmatrix}
\]

so \( T_{(x_0,y_0,z_0)}S \) is spanned by the vectors \( (1, 0, f_x(x_0, y_0)) \) and \( (0, 1, f_y(x_0, y_0)) \); i.e. all vectors \( (x, y, z) \) with \( z = f_x(x_0, y_0)x + f_y(x_0, y_0)y \). This plane is just the graph of the function sending \( (x, y) \) to \( (dx)(x_0, y_0)(x, y) \); translating it so that it passes through \( (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0)) \) yields the equation in do Carmo.

2-4.8. Since \( L \) is differentiable in a neighborhood of \( S \) — in fact on all of \( \mathbb{R}^3 \) — it follows that \( L|S : S \to \mathbb{R}^3 \) is differentiable. But \( L(S) \subset S \) by hypothesis; hence \( L|S : S \to S \) is differentiable. We also have, for \( p \in S \),

\[
(d(L|S)(p))(\alpha'(0)) = (L \circ \alpha)'(0) = L(\alpha'(0))
\]

by the chain rule and the fact that \( L \) is a linear transformation. Since \( \alpha : I \to S \) is an arbitrary curve with \( \alpha(0) = p \), we must have

\[
(d(L|S)(p))(w) = L(w)
\]

for all \( w \in T_pS \).

2-4.15. Without loss of generality, we may assume that all normals to our connected surface \( S \) pass through the origin. This means that \( v \) is orthogonal to \( T_v(S) \) for all \( v \in S \). Now let \( v_0 \in S \) and define

\[
C = \{ v \in S : |v| = |v_0| \}.
\]
Then is a differentiable function, 

Proof. Suppose that the preceding lemma shows that \( v \) is open in \( S \). Assuming this, it follows by the connectivity of \( S \) that \( C = S \); therefore \( S \) is a subset of the sphere of radius \(|v_0|\).

We now prove the claim. Suppose \( v \in S \), and let \( x : U \to N \) be a parametrization, where \( N \) is a neighborhood of \( v \) in \( S \). Without loss of generality, we may assume that \( U \) is a disk of radius \( \varepsilon > 0 \) with center at the origin and that \( x((0,0)) = v \). Let \( w \in N \). Then there exists a smooth curve \( \alpha : I \to N \) with \( \alpha(t_0) = w \) and \( \alpha(t_1) = v \), \( [t_0,t_1] \subset I \). (For example, write \( w = x(q) \), and let \( \beta \) be a parametrized curve whose trace is a line segment passing through \( q \) and \((0,0)\).

Then take \( \alpha = x \circ \beta \).

Next observe that \( \alpha(s) \cdot \alpha'(s) = 0 \) for all \( s \in I \) since \( \alpha'(s) \in T_{\alpha(s)}(S) \); this implies that \( |\alpha(s)|^2 = \alpha(s) \cdot \alpha(s) \) is constant. In particular,

\[
|w|^2 = \alpha(t_0) \cdot \alpha(t_0) = \alpha(t_1) \cdot \alpha(t_1) = |v|^2.
\]

Hence \( w \in C \), and since \( w \in N \) was arbitrary, we have that \( N \subset C \). This proves that \( C \) is open in \( S \).

2.4.17. We first prove a preliminary result.

**Lemma.** Let \( S = \{ (x,y,z) : f(x,y,z) = 0 \} \) be a regular surface, where \( f : V \to \mathbb{R} \) is a differentiable function, \( V \) open in \( \mathbb{R}^3 \). Suppose \( p \in S \) is a regular point of \( f \). Then \( (f_x(p), f_y(p), f_z(p)) \) is a non-zero normal to \( T_p S \).

**Proof.** Suppose \( \alpha : I \to S \) is a smooth curve with \( \alpha(0) = p \). Write \( \alpha(t) = (x(t), y(t), z(t)) \). Then \( f(x(t), y(t), z(t)) = 0 \) for all \( t \in I \). Differentiating at \( t = 0 \) now yields

\[
(f_x(p)x'(0) + f_y(p)y'(0) + f_z(p)z'(0) = 0:
\]

i.e.

\[
(f_x(p), f_y(p), f_z(p)) \cdot \alpha'(0) = 0.
\]

Since the set of vectors \( v \) satisfying

\[
(f_x(p), f_y(p), f_z(p)) \cdot v = 0
\]

is a plane — remember, \( (f_x(p), f_y(p), f_z(p)) \neq (0,0,0) \) as \( p \) is a regular point — we must have that this plane is \( T_p S \). This completes the proof.

Locally, any regular surface \( S \) is given by \( S = f^{-1}(0) \), where \( f : V \to \mathbb{R} \) is differentiable, \( V \) is open in \( \mathbb{R}^3 \), and 0 is a regular value. To see this, recall that locally \( S \) is the graph of a differentiable function \( h : U \to \mathbb{R} \) with \( U \) open in \( \mathbb{R}^2 \) — say, \( S = \{ (x,y,z) : z = h(x,y) \} \). Then \( S = f^{-1}(0) \), where \( f(x,y,z) = z - h(x,y) \).

Now let \( S_1 \) and \( S_2 \) be regular surfaces and let \( p \in S_1 \cap S_2 \). In a neighborhood \( V \) of \( p \) in \( \mathbb{R}^3 \), we may assume that

\[
S_1 \cap V = \{ (x,y,z) : f(x,y,z) = 0 \}
\]

\[
S_2 \cap V = \{ (x,y,z) : g(x,y,z) = 0 \}
\]

and that 0 is a regular value of both \( f \) and \( g \). Moreover, if \( q \in S_1 \cap S_2 \cap V \), the preceding lemma shows that \( (f_x(q), f_y(q), f_z(q)) \) and \( (g_x(q), g_y(q), g_z(q)) \) are non-zero normals to \( T_q(S_1) \) and \( T_q(S_2) \) respectively. But \( T_q(S_1) \neq T_q(S_2) \) by hypothesis; therefore \( (f_x(q), f_y(q), f_z(q)) \) and \( (g_x(q), g_y(q), g_z(q)) \) are linearly independent.
Finally, consider the function $F : V \rightarrow \mathbb{R}^2$ given by

$$F(x, y, z) = (f(x, y, z), g(x, y, z)).$$

$F^{-1}(0, 0) = V \cap S_1 \cap S_2$; it therefore follows from Exercise 17 of 2-2 that $V \cap S_1 \cap S_2$ is a regular curve provided that $(0, 0)$ is a regular value of $F$. But

$$(dF)(q) = \begin{bmatrix} f_x(q) & f_y(q) & f_z(q) \\ g_x(q) & g_y(q) & g_z(q) \end{bmatrix}$$

has rank 2 for $q \in F^{-1}(0, 0)$, since $(f_x(q), f_y(q), f_z(q))$ and $(g_x(q), g_y(q), g_z(q))$ are linearly independent. This proves that $(dF)(q)$ is onto and that $(0, 0)$ is a regular value.

Since the notion of regular curve is local, we therefore have that $S_1 \cap S_2$ is a regular curve.