Math 442

Winter 2019

Solutions to Homework 5

2-4.2. Let $S = \{\,(x, y, z) : x^2 + y^2 - z^2 = 1\,\}$, and suppose $\alpha : I \to S$ is a curve with $\alpha(0) = (x, y, 0)$. Write $\alpha(t) = (x(t), y(t), z(t))$. Since $x(t)^2 + y(t)^2 - z(t)^2 = 1$ for all t, we may differentiate both sides at $t = 0$ to obtain

$$2x(0)x'(0) + 2y(0)y'(0) - 2z(0)z'(0) = 0.$$

But $(x(0), y(0), z(0)) = (x, y, 0)$; therefore $\alpha'(0)$ is orthogonal to $(x, y, 0)$. Since the set of vectors orthogonal to $(x, y, 0)$ is a plane, this must be $T_{(x,y,0)}S$, and the tangent plane is parallel to the z-axis.

2-4.3. Let S be the graph of the differentiable function $z = f(x, y)$. Then $\mathbf{x} : U \to S$ defined by $\mathbf{x}(u, v) = (u, v, f(u, v))$ is a parametrization. Let $(x_0, y_0) \in U$ and $z_0 = f(x_0, y_0)$. The tangent space to S at (x_0, y_0, z_0) is given by $(d\mathbf{x})(x_0, y_0)(\mathbb{R}^2)$. But

$$(d\mathbf{x})(x_0, y_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x(x_0, y_0) & f_y(x_0, y_0) \end{bmatrix}$$

so $T_{(x_0,y_0,z_0)}S$ is spanned by the vectors $(1, 0, f_x(x_0, y_0))$ and $(0, 1, f_y(x_0, y_0))$; i.e. all vectors (x, y, z) with $z = f_x(x_0, y_0)x + f_y(x_0, y_0)y$. This plane is just the graph of the function sending (x, y) to $(df)(x_0, y_0)(x, y)$; translating it so that it passes through $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ yields the equation in do Carmo.

2-4.8. Since L is differentiable in a neighborhood of S — in fact on all of $\mathbb{R}^3$— it follows that $L|S : S \to \mathbb{R}^3$ is differentiable . But $L(S) \subset S$ by hypothesis; hence $L|S : S \to S$ is differentiable. We also have, for $p \in S$,

$$(d(L|S)(p))(\alpha'(0)) = (L \circ \alpha)'(0) = L(\alpha'(0))$$

by the chain rule and the fact that L is a linear transformation. Since $\alpha : I \to S$ is an arbitrary curve with $\alpha(0) = p$, we must have

$$(d(L|S)(p))(w) = L(w)$$

for all $w \in T_pS$.

2-4.15. Without loss of generality, we may assume that all normals to our connected surface S pass through the origin. This means that v is orthogonal to $T_v(S)$ for all $v \in S$. Now let $v_0 \in S$ and define

$$C = \{\, v \in S : |v| = |v_0|\,\}.$$

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C is non-empty (since $v_0 \in C$) and is closed in S, since $C = f^{-1}(|v_0|)$, where $f : S \to \mathbb{R}$ is the continuous function defined by $f(v) = |v|$. I claim that C is open in S. Assuming this, it follows by the connectivity of S that $C = S$; therefore S is a subset of the sphere of radius $|v_0|$.

We now prove the claim. Suppose $v \in S$, and let $\mathbf{x} : U \to N$ be a parametrization, where N is a neighborhood of v in S. Without loss of generality, we may assume that U is a disk of radius $\varepsilon > 0$ with center at the origin and that $\mathbf{x}((0,0)) = v$. Let $w \in N$. Then there exists a smooth curve $\alpha : I \to N$ with $\alpha(t_0) = w$ and $\alpha(t_1) = v$, $[t_0, t_1] \subset I$. (For example, write $w = \mathbf{x}(q)$, and let β be a parametrized curve whose trace is a line segment passing through q and $(0,0)$. Then take $\alpha = \mathbf{x} \circ \beta$.)

Next observe that $\alpha(s) \cdot \alpha'(s) = 0$ for all $s \in I$ since $\alpha'(s) \in T_{\alpha(s)}(S)$; this implies that $|\alpha(s)|^2 = \alpha(s).\alpha(s)$ is constant. In particular,

$$|w|^2 = \alpha(t_0) \cdot \alpha(t_0) = \alpha(t_1) \cdot \alpha(t_1) = |v|^2.$$

Hence $w \in C$, and since $w \in N$ was arbitrary, we have that $N \subset C$. This proves that C is open in S.

2-4.17. We first prove a preliminary result.

Lemma. *Let $S = \{ (x,y,z) : f(x,y,z) = 0 \}$ be a regular surface, where $f : V \to \mathbb{R}$ is a differentiable function, V open in $\mathbb{R}^3$. Suppose $p \in S$ is a regular point of f. Then $(f_x(p), f_y(p), f_z(p))$ is a non-zero normal to T_pS.*

Proof. Suppose $\alpha : I \to S$ is a smooth curve with $\alpha(0) = p$. Write $\alpha(t) = (x(t), y(t), z(t))$. Then $f(x(t), y(t), z(t)) = 0$ for all $t \in I$. Differentiating at $t = 0$ now yields

$$f_x(p)x'(0) + f_y(p)y'(0) + f_z(p)z'(0) = 0 :$$

i.e.

$$(f_x(p), f_y(p), f_z(p)) \cdot \alpha'(0) = 0.$$

Since the set of vectors v satisfying

$$(f_x(p), f_y(p), f_z(p)) \cdot v = 0$$

is a plane — remember, $(f_x(p), f_y(p), f_z(p)) \neq (0,0,0)$ as p is a regular point — we must have that this plane is T_pS. This completes the proof.

Locally, any regular surface S is given by $S = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is differentiable, V is open in $\mathbb{R}^3$, and 0 is a regular value. To see this, recall that locally S is the graph of a differentiable function $h : U \to \mathbb{R}$ with U open in $\mathbb{R}^2$ —say, $S = \{ (x,y,z) : z = h(x,y) \}$. Then $S = f^{-1}(0)$, where $f(x,y,z) = z - h(x,y)$.

Now let S_1 and S_2 be regular surfaces and let $p \in S_1 \cap S_2$. In a neighborhood V of p in $\mathbb{R}^3$, we may assume that

$$S_1 \cap V = \{ (x,y,z) : f(x,y,z) = 0 \}$$
$$S_2 \cap V = \{ (x,y,z) : g(x,y,z) = 0 \}$$

and that 0 is a regular value of both f and g. Moreover, if $q \in S_1 \cap S_2 \cap V$, the preceding lemma shows that $(f_x(q), f_y(q), f_z(q))$ and $(g_x(q), g_y(q), g_z(q))$ are non-zero normals to $T_q(S_1)$ and $T_q(S_2)$ respectively. But $T_q(S_1) \neq T_q(S_2)$ by hypothesis; therefore $(f_x(q), f_y(q), f_z(q))$ and $(g_x(q), g_y(q), g_z(q))$ are linearly independent.

Finally, consider the function $F : V \to \mathbb{R}^2$ given by

$$F(x, y, z) = (f(x, y, z), g(x, y, z)).$$

$F^{-1}(0,0) = V \cap S_1 \cap S_2$; it therefore follows from Exercise 17 of 2-2 that $V \cap S_1 \cap S_2$ is a regular curve provided that $(0,0)$ is a regular value of F. But

$$(dF)(q) = \begin{bmatrix} f_x(q) & f_y(q) & f_z(q) \\ g_x(q) & g_y(q) & g_z(q) \end{bmatrix}$$

has rank 2 for $q \in F^{-1}(0,0)$, since $(f_x(q), f_y(q), f_z(q))$ and $(g_x(q), g_y(q), g_z(q))$ are linearly independent. This proves that $(dF)(q)$ is onto and that $(0,0)$ is a regular value.

Since the notion of regular curve is local, we therefore have that $S_1 \cap S_2$ is a regular curve.