Math 442

Winter 2019

Solutions to Homework 4

2-3.2. The map $\pi : S \to \mathbb{R}^2$ extends to a smooth function from all of $\mathbb{R}^3$ to $\mathbb{R}^2$; namely, the map which takes (x, y, z) to (x, y). This implies by a result proved in lecture (or see example 3 of 2-3), that π is smooth.

2-3.14. Certainly, if A is open in S, then A is a regular surface.

Conversely, suppose that $A \subset S$ and that A is a regular surface. To show that A is open in S, it suffices to show that whenever $p \in A$, there exists a neighborhood of p in S completely contained in A. To do this, start by taking $\mathbf{x} : U \to N$ to be a parametrization of A in a neighborhood of p and $\mathbf{y} : U' \to N'$ to be a parametrization of S in a neighborhood of p. By shrinking U' if necessary, we may, as in the proof of Proposition 1 of this section, extend $\mathbf{y}$ to a diffeomorphism $\mathbf{Y} : V \to W$, where V and W are open in $\mathbb{R}^3$. (Here we regard $\mathbb{R}^2 \subset \mathbb{R}^3$ in the usual way.) Then, by shrinking U if necessary, we get that

$$\mathbf{y}^{-1} \circ \mathbf{x} = \mathbf{Y}^{-1} \circ \mathbf{x} : U \to \mathbb{R}^2$$

is differentiable. Moreover, for $q \in U$,

$$d(\mathbf{y}^{-1} \circ \mathbf{x})(q) = (d\mathbf{Y}^{-1})(\mathbf{x}(q)) \circ d\mathbf{x}(q)$$

is the composition of two one-to-one linear transformations and so is one-to-one. Note that we have written $d(\mathbf{y}^{-1} \circ \mathbf{x})(q)$ as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. However, $\mathbf{y}^{-1} \circ \mathbf{x}$ takes all of U to $\mathbb{R}^2$. This implies that the last row of the matrix representing $d(\mathbf{Y}^{-1})(\mathbf{x}(q)) \circ d\mathbf{x}(q)$ must be zero and that it must therefore be a linear transformation whose image lies in $\mathbb{R}^2$. Since it is one-to-one with domain $\mathbb{R}^2$, it must be an isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. This holds for all $q \in U$; it therefore follows by the inverse function theorem that $(\mathbf{y}^{-1} \circ \mathbf{x})(U)$ is open in $\mathbb{R}^2$. But $\mathbf{y}^{-1}$ is a homeomorphism from an open subset of S to an open subset of $\mathbb{R}^2$, so that $N = \mathbf{x}(U)$ is open in S. This completes the proof.