Math 442
Winter 2019
Solutions to Homework 4

2-3.2. The map $\pi : S \to \mathbb{R}^2$ extends to a smooth function from all of $\mathbb{R}^3$ to $\mathbb{R}^2$; namely, the map which takes $(x, y, z)$ to $(x, y)$. This implies by a result proved in lecture (or see example 3 of 2-3), that $\pi$ is smooth.

2-3.14. Certainly, if $A$ is open in $S$, then $A$ is a regular surface. Conversely, suppose that $A \subset S$ and that $A$ is a regular surface. To show that $A$ is open in $S$, it suffices to show that whenever $p \in A$, there exists a neighborhood of $p$ in $S$ completely contained in $A$. To do this, start by taking $x : U \to N$ to be a parametrization of $A$ in a neighborhood of $p$ and $y : U' \to N'$ to be a parametrization of $S$ in a neighborhood of $p$. By shrinking $U'$ if necessary, we may, as in the proof of Proposition 1 of this section, extend $y$ to a diffeomorphism $Y : V \to W$, where $V$ and $W$ are open in $\mathbb{R}^3$. (Here we regard $\mathbb{R}^2 \subset \mathbb{R}^3$ in the usual way.) Then, by shrinking $U$ if necessary, we get that

$$y^{-1} \circ x = Y^{-1} \circ x : U \to \mathbb{R}^2$$

is differentiable. Moreover, for $q \in U$,

$$d(y^{-1} \circ x)(q) = (dY^{-1})(x(q)) \circ dx(q)$$

is the composition of two one-to-one linear transformations and so is one-to-one. Note that we have written $d(y^{-1} \circ x)(q)$ as a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. However, $y^{-1} \circ x$ takes all of $U$ to $\mathbb{R}^2$. This implies that the last row of the matrix representing $d(Y^{-1})(x(q)) \circ dx(q)$ must be zero and that it must therefore be a linear transformation whose image lies in $\mathbb{R}^2$. Since it is one-to-one with domain $\mathbb{R}^2$, it must be an isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. This holds for all $q \in U$; it therefore follows by the inverse function theorem that $(y^{-1} \circ x)(U)$ is open in $\mathbb{R}^2$. But $y^{-1}$ is a homeomorphism from an open subset of $S$ to an open subset of $\mathbb{R}^2$, so that $N = x(U)$ is open in $S$. This completes the proof.