Math 442
Homework #3
Solutions

2.2.1 Consider the function $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^2+y^2$. $p=(x,y,z)$ is a critical point of f if and only if $x=0$ and $y=0$. Thus 1 is not a critical value, so

$$f^{-1}\{1\} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1\}$$

is a regular surface.

Let

$$N_1 = f^{-1}\{1\} \cap \{(x,y,z) : x>0\}$$
$$N_2 = f^{-1}\{1\} \cap \{(x,y,z) : x<0\}$$
$$N_3 = f^{-1}\{1\} \cap \{(x,y,z) : y>0\}$$
$$N_4 = f^{-1}\{1\} \cap \{(x,y,z) : y<0\}.$$

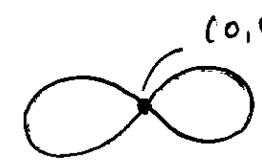
Each N_i is a coordinate neighborhood; for example, let $U_4 = \{(x,z) : |x|<1\}$. Then $\underline{x}_4 : U_4 \to N_4$ defined by $\underline{x}_4(x,z) = (x, -\sqrt{1-x^2}, z)$ is a parametrization. Moreover, $N_1 \cup N_2 \cup N_3 \cup N_4 = f^{-1}\{1\}$.

2.2.2 The set $S_2 = \{(x,y,z) : z=0 \text{ and } x^2+y^2<1\}$ is a regular surface since $\underline{x}: U \to S_2$ defined by $\underline{x}(u,v) = (u,v,0)$, where $U = \{(u,v) ; u^2+v^2<1\}$, is a parametrization.

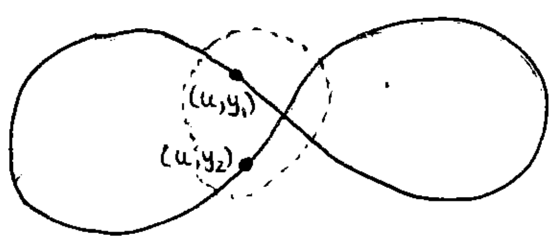
On the other hand, $S_1 = \{(x,y,z) : z=0 \text{ and } x^2+y^2 \leq 1\}$ is not a regular surface. Consider, for example, the point $p=(1,0,0)$, and suppose there exists a parametrization $\underline{x}: U \to S_1 \cap V$ with $p \in V$. Since the z-component coordinate function for $\underline{x}$ is the 0 function, it follows that $d(\pi \circ \underline{x})(q)$ is invertible for any $q \in U$, where $\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by $\pi(x,y,z) = (x,y)$. It then follows by the inverse function theorem that $(\pi \circ \underline{x})(U)$ must be open in $\mathbb{R}^2$. But $(\pi \circ \underline{x})(U) = \pi(S_1 \cap V)$ is not open in $\mathbb{R}^2$, since no open ball in $\mathbb{R}^2$ around $(1,0) = \pi(1,0,0)$ is contained in $\pi(S_1) = \{(x,y): x^2+y^2 \leq 1\}$. This proves that no such parametrization $\underline{x}$ can exist, so S_1 is not a regular surface.

2.2.4. $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 2z$; therefore $(x,y,0)$ is a critical point of f for any $x,y \in \mathbb{R}$. Since $f(x,y,0) = 0$, 0 is not a regular value of f. Nevertheless, $f^{-1}\{0\} = \{(x,y,z): z=0\}$ is a regular surface, since $\underline{x}: \mathbb{R}^2 \longrightarrow f^{-1}\{0\}$ defined by $\underline{x}(u,v) = (u,v,0)$ is a parametrization.

10. S is not a regular surface; we will show that there is no coordinate neighborhood N containing $(0,0,0)$, where C is configured as shown:



Indeed, if N existed, then we could assume, by shrinking N if necessary, that N was the graph of a smooth function g defined on an open set U in $\mathbb{R}^2$. Certainly we can't have $N = \{(u, v, g(u,v)) : (u,v) \in U\}$, since there are, for example, many z such that $(0,0,z) \in N$. We also can't have $N = \{(u, g(u,v), v) : (u,v) \in U\}$, since, as the following picture shows, for each small u, $u \neq 0$, there exist y_1, y_2 with $y_1 \neq y_2$ but $(u, y_1, 0) \in N$ and $(u, y_2, 0) \in N$. Similarly, we can't have $N = \{(g(u,v), u, v) : (u,v) \in U\}$; this therefore shows that S is not a regular surface.



16. a. The line segment from $(u,v,0)$ to $(0,0,2)$ is given by $\{t(u,v,0)+(1-t)(0,0,2) : 0 \le t \le 1\}$. First find the value of $t \neq 0$ such that $t(u,v,0)+(1-t)(0,0,2)$ intersects the sphere. At this value of t, we have

$$(tu)^2+(tv)^2+[2(1-t)-1]^2=1.$$

Solving this we get

$$t(tu^2+tv^2+4t-4)=0,$$

so

$$tu^2+tv^2+4t-4=0.$$

Hence $t=\frac{4}{u^2+v^2+4}$, and the point of intersection is

$$(tu,\ tv,\ 2-2t) = \left(\frac{4u}{u^2+v^2+4},\ \frac{4v}{u^2+v^2+4},\ \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

This proves that $\pi^{-1}: \mathbb{R}^2 \longrightarrow S^2-\{N\}$ is given by

$$\pi^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4},\ \frac{4v}{u^2+v^2+4},\ \frac{2(u^2+v^2)}{u^2+v^2+4}\right).$$

b. We know that the northern hemisphere is a coordinate neighborhood; it thus suffices to show that $\pi^{-1}: \mathbb{R}^2 \longrightarrow S^2-\{N\}$ is a parametrization. To do this, first observe that π is given by $\pi(x,y,z) = \left(\frac{2x}{2-z}, \frac{2y}{2-z}\right)$ – I'll leave the verification to you — this proves that π^{-1} is a homeomorphism. Clearly, π^{-1} is smooth; we need thus only show that $(d\pi^{-1})(u,v)$ is one-to-one for any $(u,v) \in \mathbb{R}^2$. To do this, observe that we may define

$$\pi : \mathbb{R}^3 - \{(x,y,z) : z = 2\} \longrightarrow \mathbb{R}^2$$

by the formula above. Then π is differentiable, and we have $\pi \circ \pi^{-1} = id_{\mathbb{R}^2}$. It now follows by the chain rule that

$$(d\pi)(\pi^{-1}(u,v)) \circ (d\pi^{-1})(u,v) = I$$

where I is the identity linear transformation on $\mathbb{R}^2$. Hence $(d\pi)^{-1}(u,v)$ has a left inverse and so must be one-to-one. This completes the proof.

17. A set C in $\mathbb{R}^n$ is a regular curve if for each $p \in C$, there exists an open set I in $\mathbb{R}$, an open neighborhood V of p in $\mathbb{R}^n$, and a homeomorphism $\underline{x}: I \longrightarrow V \cap C$ such that

i. $\underline{x}: I \longrightarrow \mathbb{R}^n$ is smooth

ii. $\underline{x}'(t) \neq \vec{0}$ for each $t \in I$. (This is the same as saying that $(d\underline{x})(t)$ is one-to-one.)

a. Let $C = f^{-1}\{a\}$ where a is a regular value of f, and suppose $p = (x_0, y_0) \in C$. Assume that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. (The case $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$ is similar.) Consider the function $F: U \to \mathbb{R}^2$ defined by

$$F(x,y) = (x, f(x,y)).$$

Then

$$\det(dF(p)) = \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{vmatrix} = \frac{\partial f}{\partial y}(p) \neq 0,$$

so $dF(p)$ is invertible. By the inverse function theorem, there then exists a neighborhood V of p in $\mathbb{R}^2$ such that $W = F(V)$ is open in $\mathbb{R}^2$ and $F: V \to W$ is a diffeomorphism. Observe that $F^{-1}: W \longrightarrow V$

restricts to a homeomorphism from $W \cap \{(u,v) : v=a\}$ to $V \cap C$. Let $I = \{u \mid (u,a) \in W\}$. Then $j : I \longrightarrow W \cap \{(u,v) : v=a\}$ given by $j(u) = (u,a)$ is a homeomorphism, so

$$F^{-1} \circ j : I \longrightarrow V \cap C$$

is a homeomorphism as well. Since F^{-1} and j are both differentiable, so is $F^{-1} \circ j$. Finally,

$$d(F^{-1} \circ j)(t) = dF^{-1}(j(t)) \circ dj(t)$$

and both $dF^{-1}(j(t))$ and $dj(t)$ are one-to-one. This proves that $d(F^{-1} \circ j)(t)$ is one-to-one for each $t \in I$ and therefore $F^{-1} \circ j$ is a parametrization. (Observe here that $F^{-1}(u,v) = (u, g(u,v))$ for some smooth function $g : W \longrightarrow \mathbb{R}$. Then

$$(F^{-1} \circ j)(t) = (t, g(t,a)),$$

so that $V \cap C$ is the graph of the smooth function whose value at t is $g(t,a)$.)

$f^{-1}\{a\}$ need not be connected. For example, let $f(x,y) = x^2$. Then 1 is a regular value, but $f^{-1}\{1\}$ consists of the lines $x=1$ and $x=-1$ in $\mathbb{R}^2$.

b. Let $a = (a_1, a_2)$ be a regular value of F, and let $C = F^{-1}\{a\} \subset \mathbb{R}^3$. Suppose $p = (x_0, y_0, z_0) \in C$. Then

$$dF(p) = \begin{bmatrix} \frac{\partial F_1}{\partial x}(p) & \frac{\partial F_1}{\partial y}(p) & \frac{\partial F_1}{\partial z}(p) \\ \frac{\partial F_2}{\partial x}(p) & \frac{\partial F_2}{\partial y}(p) & \frac{\partial F_2}{\partial z}(p) \end{bmatrix}$$

has rank 2. This means that at least one of the Jacobian determinants

$$\frac{\partial(F_1, F_2)}{\partial(x,y)}, \quad \frac{\partial(F_1, F_2)}{\partial(x,z)}, \quad \frac{\partial(F_1, F_2)}{\partial(y,z)}$$

must be nonzero at p. Let us suppose that $\frac{\partial(F_1,F_2)}{\partial(y,z)}(p) \neq 0$; the other cases are similar.

Consider the function $G: U \longrightarrow \mathbb{R}^3$ defined by

$$G(x,y,z) = (x, F(x,y,z)).$$

Then

$$\det(dG(p)) = \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial F_1}{\partial x}(p) & \frac{\partial F_1}{\partial y}(p) & \frac{\partial F_1}{\partial z}(p) \\ \frac{\partial F_2}{\partial x}(p) & \frac{\partial F_2}{\partial y}(p) & \frac{\partial F_2}{\partial z}(p) \end{vmatrix} = \frac{\partial(F_1,F_2)}{\partial(y,z)}(p) \neq 0;$$

hence, by the inverse function theorem, there exists a neighborhood V of p in $\mathbb{R}^3$ such that $W = G(V)$ is open in $\mathbb{R}^3$ and $G : V \to W$ is a diffeomorphism.

As before,

$$G^{-1} : W \cap \{(u,v,w) : (v,w) = (a_1,a_2)\} \longrightarrow V \cap C$$

is a homeomorphism. Now let $I = \{u : (u,a_1,a_2) \in W\}$. Then $j : I \longrightarrow W \cap \{(u,v,w) : (v,w) = (a_1,a_2)\}$ given by $j(u) = (u,a_1,a_2)$ is a homeomorphism, so

$$G^{-1} \circ j : I \longrightarrow V \cap C$$

is a homeomorphism as well. The same argument as in part a) shows that $G^{-1} \circ j$ is differentiable and that $d(G^{-1} \circ j)(t)$ is one-to-one for each $t \in I$; this then proves that $G^{-1} \circ j$ is a parametrization. Observe also that, as before,

$$(G^{-1} \circ j)(t) = (t, h(t))$$

for some smooth function $h : I \to \mathbb{R}^2$. Thus $V \cap C$ is the graph of the function h.

Finally, note that $C = S_1 \cap S_2$, where $S_1 = F_1^{-1}\{a_1\}$ and $S_2 = F_2^{-1}\{a_2\}$. However, it need not be the case that a_1 is a regular value of F_1 and a_2 is a regular value of F_2, so we cannot conclude that S_1 and S_2 are always regular surfaces.

c. We first prove a preliminary result.

Lemma: Suppose C is a regular curve in $\mathbb{R}^2$ and $p \in C$. Then there exists a neighborhood V of p in C such that V is the graph of a differentiable function of the form $y = f(x)$ or $x = g(y)$.

Proof: Let $\underline{x} : U_0 \longrightarrow V_0$ be a parametrization, with U_0 open in $\mathbb{R}$, V_0 open in C, $p \in V_0$, and $\underline{x}(t_0) = p$. Write $\underline{x}(t) = (x(t), y(t))$. Since $\underline{x}'(t_0) \neq \vec{0}$, we must have either $x'(t_0) \neq 0$ or $y'(t_0) \neq 0$. Suppose $x'(t_0) \neq 0$; the other case is similar. Then there exists a neighborhood U of t_0 such that $W = x(U)$ is open in $\mathbb{R}$ and $x : U \longrightarrow W$ is a diffeomorphism. Let $V = \underline{x}(U)$. Since $\underline{x}$ is a homeomorphism and V_0 is open in C, it follows that V is a neighborhood of p in C. Moreover, $x = \pi \circ \underline{x}$, so $\pi : V \longrightarrow W$ is also a homeomorphism, where π is the projection onto the first coordinate. Therefore, given $t \in W$, there exists a unique $s \in \mathbb{R}$ such that $(t, s) \in V$. Define $f : W \longrightarrow \mathbb{R}$ by $f(t) = s$. Then V is the graph of f, and f is smooth since $f = \pi_2 \circ \underline{x} \circ x^{-1}$, where π_2 is the projection onto the second coordinate. This completes the proof.

Now let $C = \{(x,y) \in \mathbb{R}^2 : x^2 = y^3\}$. (See Figure 1-2 on p. 3 of do Carmo for a picture.) I claim that there is no neighborhood V of $(0,0)$ in C which is the graph of a differentiable function $y = f(x)$ or $x = g(y)$. Certainly, we cannot have $x = g(y)$, since both $(y^{3/2}, y)$ and $(-y^{3/2}, y)$ lie in V for y a sufficiently small positive number. (Alternatively, you could observe that there are no points in C with $y < 0$, so g could not be defined in a neighborhood of 0.) If V is the graph of a function $y = f(x)$, we must have $f(x) = x^{2/3}$. But f is not differentiable at 0.