**Math 442**

**Winter 2019**

**Solutions to Homework 2**

1-5.1. a. $d\alpha/ds = (-(a/c)\sin(s/c), (a/c)\cos(s/c), b/c)$ and thus

$$\begin{aligned}|\alpha'(s)| &= \sqrt{\frac{a^2}{c^2}\sin^2\frac{s}{c} + \frac{a^2}{c^2}\cos^2\frac{s}{c} + \frac{b^2}{c^2}} \\ &= \sqrt{\frac{a^2+b^2}{c^2}} = 1,\end{aligned}$$

since $c^2 = a^2 + b^2$. Therefore $\alpha$ is parametrized by arc length.

b. $\alpha''(s) = (-(a/c^2)\cos(s/c), -(a/c^2)\sin(s/c), 0)$ so $k(s) = |\alpha''(s)| = a/c^2$. To compute the torsion, start with

$$\begin{aligned}b(s) = t(s)\wedge n(s) &= \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \wedge \frac{c^2}{a}\left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right) \\ &= \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \wedge \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right) \\ &= \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right).\end{aligned}$$

Then $b'(s) = ((b/c^2)\cos(s/c), b/c^2\sin(s/c), 0) = -(b/c^2)n(s)$, so that $\tau(s) = -b/c^2$.

c. The plane passing through $\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c)$ with normal $b(s) = ((b/c)\sin(s/c), -(b/c)\cos(s/c), a/c)$ is the osculating plane; it is given by the set of $(x, y, z)$ with

$$\frac{b}{c}\sin\frac{s}{c}\cdot\left(x - a\cos\frac{s}{c}\right) - \frac{b}{c}\cos\frac{s}{c}\cdot\left(y - a\sin\frac{s}{c}\right) + \frac{a}{c}\left(z - \frac{bs}{c}\right) = 0.$$

Simplify this expression to get

$$\left(\frac{b}{c}\sin\frac{s}{c}\right)x - \left(\frac{b}{c}\cos\frac{s}{c}\right)y + \frac{a}{c}z = \frac{bas}{c^2}.$$

d. Since $n(s) = (-\cos(s/c), -\sin(s/c), 0)$, it follows that the line through $\alpha(s)$ in the direction of $n(s)$ is parallel to the $xy$-plane. It also intersects the $z$-axis at the point $(0, 0, bs/c)$ and must therefore do so at an angle of $\pi/2$.

e. $e_3\cdot(\alpha'(s)/|\alpha'(s)| = \cos\theta$, where $\theta$ is the angle between $\alpha'(s)$ and the $z$-axis. Since

$$e_3\cdot\frac{\alpha'(s)}{|\alpha'(s)|} = \frac{b}{c},$$

it follows that this angle is constant.

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1-5.5. a. Without loss of generality, we may assume that $\alpha$ is parametrized by arc length; we may also assume, by translating if necessary, that the fixed point through which all the tangent lines pass is the origin. This implies that the line through the origin and $\alpha(s)$ is the tangent line of $\alpha$ at $s$; hence

$$\alpha(s) = c(s)\alpha'(s) \tag{1}$$

for some scalar $c(s)$. In fact, $c(s) = \alpha(s) \cdot \alpha'(s)$, so that $c$ is a differentiable function of $s$. Now differentiate both sides of (1) and rearrange to get

$$c(s)\alpha''(s) = (1 - c'(s))\alpha'(s).$$

But $\alpha''(s)$ is perpendicular to $\alpha'(s)$ (since $\alpha$ is parametrized by arc length); the above formula then implies that $c(s)\alpha''(s) = 0$ for all $s$. I claim that in fact $\alpha''(s) = 0$ for all $s$ and hence, by a previous homework exercise (1-2.3), the trace of $\alpha$ is a straight line (segment). Indeed, let $s$ be in the domain of $\alpha$. If $c(s) \neq 0$, then $\alpha''(s) = 0$. If $c(s) = 0$, then $\alpha(s) = 0$. But $\alpha'(s) \neq 0$, so by the definition of the derivative, it follows that there exists an $\epsilon > 0$ such that $\alpha(t) \neq 0$ whenever $|t - s| < \epsilon$ and $t \neq s$. This implies that $c(t) \neq 0$ and hence $\alpha''(t) = 0$ for all such $t$. But $\alpha''$ is continuous; therefore $\alpha''(s) = 0$ as well. This completes the proof.

b. If $\alpha$ is not regular, the conclusion of part a) need not hold. For example, consider $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = \begin{cases} e^{-\frac{1}{t^2}}(t, -t) & t < 0 \\ (0,0) & t = 0 \\ e^{-\frac{1}{t^2}}(t, t) & t > 0. \end{cases}$$

(The factor $e^{-\frac{1}{t^2}}$ is introduced so that $\alpha$ is (infinitely) differentiable. Recall that the function

$$f(t) = \begin{cases} 0 & t = 0 \\ e^{-\frac{1}{t^2}} & t \neq 0 \end{cases}$$

is an infinitely differentiable function on $\mathbb{R}$ all of whose derivatives vanish at 0.) However, the trace of $\alpha$ is the graph of the function $y = |x|$ for $|x| < 1$, yet all of the tangent lines of $\alpha$ pass through the origin.

1-5.6. a. Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation, and let $u$, $v$ be elements of $\mathbb{R}^3$. We have

$$|u|^2 = u \cdot u = \rho(u) \cdot \rho(u) = |\rho(u)|^2,$$

so the norm of a vector is invariant under an orthogonal transformation. Similarly, let $\theta_0$ be the angle between $u$ and $v$, and let $\theta_1$ be the angle between $\rho(u)$ and $\rho(v)$, $0 \leq \theta_i \leq \pi$. Then

$$|u||v| \cos\theta_0 = u \cdot v = \rho(u) \cdot \rho(v) = |\rho(u)||\rho(v)| \cos\theta_1.$$

Since $|u| = |\rho(u)|$ and $|v| = |\rho(v)|$, we have that $\cos\theta_0 = \cos\theta_1$ (since we must have $u$ and $v$ nonzero to talk about the angle between them) and hence $\theta_0 = \theta_1$.

b. Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant and let $u$,$v \in R^3$. If $u$ and $v$ are collinear, then $u \wedge v = 0$ and hence $\rho(u \wedge v) = 0$. On the other hand, $\rho(u)$ and $\rho(v)$ are also collinear, so $\rho(u) \wedge \rho(v) = 0 = \rho(u \wedge v)$.

Now suppose that $u$ and $v$ are not collinear. Then $\rho(u)$ and $\rho(v)$ are not collinear (why?), and $\rho(u) \wedge \rho(v)$ is the unique vector of length $|\rho(u)||\rho(v)| \sin\theta_1|$ perpendicular to both $\rho(u)$ and $\rho(v)$ and such that $(\rho(u), \rho(v), \rho(u) \wedge \rho(v))$ is positively oriented. Once again, $\theta$ is the angle between $\rho(u)$ and $\rho(v)$. We will show that $\rho(u \wedge v)$ has these same properties. Indeed, $u \wedge v$ is perpendicular to both $u$ and $v$; hence $\rho(u \wedge v)$ is perpendicular to $\rho(u)$ and $\rho(v)$. Since $(u, v, u \wedge v)$ is positively oriented and $\rho$ has positive determinant, so is $(\rho(u), \rho(v), \rho(u \wedge v))$. Moreover, by part a,

$$|\rho(u \wedge v)| = |u \wedge v| = |u||v|| \sin\theta_0| = |\rho(u)||\rho(v)|| \sin\theta_1|;$$

therefore $\rho(u \wedge v) = \rho(u) \wedge \rho(v)$.

This proof also shows that if $\rho$ is orthogonal with negative determinant, then $\rho(u) \wedge \rho(v) = -\rho(u \wedge v)$.

c. It is trivial to check that the arc length, curvature, and torsion are invariant under a translation.

Now let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation with positive determinant, and let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Consider also the curve $\rho \circ \alpha : I \to \mathbb{R}^3$. Since $\rho$ is a linear transformation, it follows that

$$(\rho \circ \alpha)'(t) = \rho(\alpha'(t)), \tag{1}$$

so

$$|(\rho \circ \alpha'(t)| = |\rho(\alpha'(t))| = |\alpha'(t)| \tag{2}$$

by the orthogonality of $\rho$. It is clear from this that the arc length of $\rho \circ \alpha$ from $t_0$ to $t$ is the same as the arc length of $\alpha$ from $t_0$ to $t$. Moreover, if $\alpha$ is parametrized by arc length, so is $\rho \circ \alpha$. In such a case, $k(s)$, the curvature of $\alpha$, is given by

$$k(s) = |\alpha''(s)|,$$

and $k_\rho(s)$, the curvature of $\rho \circ \alpha$, is given by

$$k_\rho(s) = |(\rho \circ \alpha)''(s)|.$$

But $(\rho \circ \alpha)' = \rho \circ \alpha'$, so the same arguments as in (1) and (2) show that

$$(\rho \circ \alpha)''(s) = \rho(\alpha''(s))$$

and

$$|(\rho \circ \alpha)''(s)| = |\rho(\alpha''(s))| = |\alpha''(s)|.$$

Thus $k(s) = k_\rho(s)$.

Finally, let $t(s)$, $n(s)$, $b(s)$, and $\tau(s)$ be defined for $\alpha$ as in do Carmo; and let $t_\rho(s)$, $n_\rho(s)$, $b_\rho(s)$, and $\tau_\rho(s)$ be the corresponding functions for $\rho \circ \alpha$. We have already shown that

$$t_\rho(s) = \rho(t(s))$$
$$n_\rho(s) = \rho(n(s));$$

hence, by part a),

$$b_\rho(s) = t_\rho(s) \wedge n_\rho(s) = \rho(t(s)) \wedge \rho(n(s)) = \rho(t(s) \wedge n(s)) = \rho(b(s)).$$

It then follows as in (1) that

$$b'_\rho(s) = \rho(b'(s)),$$

so that

$$b'_\rho(s) = \rho(\tau(s)n(s)) = \tau(s)\rho(n(s)) = \tau(s)n_\rho(s)$$

and therefore $\tau_\rho(s) = \tau(s)$.

1-5.9. Let $\alpha(s)$ be as stated in the problem. Then $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$, so $\alpha$ is parametrized by arc length. Moreover,

$$\alpha''(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = k(s)(-\sin\theta(s), \cos\theta(s)),$$

and $(\cos\theta(s), \sin\theta(s)), (-\sin\theta(s), \cos\theta(s))$ is positively oriented. Thus the curvature of $\alpha$ at $s$ is $k(s)$.

Now suppose $\beta : I \to \mathbb{R}^2$ is a curve parametrized by arc length whose curvature is the given function $k(s)$. Since $|\beta'(s)| = 1$, we may follow the hint to write

$$\beta'(s) = (\cos\psi(s), \sin\psi(s))$$

for some differentiable function $\psi : I \to \mathbb{R}$. Then

$$\beta''(s) = \psi'(s)(-\sin\psi(s), \cos\psi(s)),$$

and since $(\cos\psi(s), \sin\psi(s)), (-\sin\psi(s), \cos\psi(s))$ is positively oriented, it follows that $\psi'(s) = k(s)$. Thus

$$\psi(s) = \int k(s)\, ds + \omega$$

for some constant $\omega$. Moreover, $\beta$ is an antiderivative of $\beta'$, so

$$\beta(s) = \left( \int \cos\psi(s)\, ds + u, \int \sin\psi(s)\, ds + v \right)$$

for some constants $u$ and $v$.

If $u$ and $v$ are changed, the curve $\beta$ is translated, and if $\omega$ is changed, $\beta'$ is rotated. This implies that its antiderivative is rotated (and perhaps translated).

1-5.12. Define $\phi(t) = \int_{t_0}^t |\alpha'(u)|\, du$ and $\beta = \alpha \circ \phi^{-1}$.

a. This problem asks you to compute $d(\phi^{-1})/ds$ and $d^2(\phi^{-1})/ds^2$. First of all,

$$\frac{d\phi^{-1}}{ds} = \frac{1}{\phi'(\phi^{-1}(s))} = \frac{1}{|\alpha'(\phi^{-1}(s))|}.$$

Now, since $|\alpha'(t)| = (\alpha'(t) \cdot \alpha'(t))^{\frac{1}{2}}$, we have

$$\frac{d}{dt}(|\alpha'(t)|) = \frac{\alpha'(t) \cdot \alpha''(t)}{|\alpha'(t)|},$$

so

$$\frac{d}{dt}\left(\frac{1}{|\alpha'(t)|}\right) = -\frac{\alpha'(t)\cdot\alpha''(t)}{|\alpha'(t)|^3}.$$

Therefore,

$$\frac{d^2\phi^{-1}}{ds^2} = -\frac{\alpha'(\phi^{-1}(s))\cdot\alpha''(\phi^{-1}(s))}{|\alpha'(\phi^{-1}(s))|^3}\cdot\frac{d\phi^{-1}}{ds} = -\frac{\alpha'(\phi^{-1}(s))\cdot\alpha''(\phi^{-1}(s))}{|\alpha'(\phi^{-1}(s))|^4}.$$

b. We need to compute the curvature of $\beta$ at $s$, where $t = \phi^{-1}(s)$. Now

$$\beta'(s) = \alpha'(\phi^{-1}(s))\cdot\frac{d\phi^{-1}}{ds} = \frac{\alpha'(t)}{|\alpha'(t)|}.$$

Hence

$$\begin{aligned}\beta''(s) &= \frac{d}{dt}\left(\frac{\alpha'(t)}{|\alpha'(t)|}\right)\frac{dt}{ds}\\ &= \left(\frac{\alpha''(t)}{|\alpha'(t)|} - \frac{(\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^3}\right)\cdot\frac{1}{|\alpha'(t)|}\\ &= \frac{(\alpha'(t)\cdot\alpha'(t))\alpha''(t) - (\alpha'(t)\cdot\alpha''(t))\alpha'(t)}{|\alpha'(t)|^4}.\end{aligned}$$

In this expression, the dot product of the numerator with itself is given by

$$|\alpha''(t)|^2|\alpha'(t)|^4 - 2(\alpha'(t)\cdot\alpha''(t))^2|\alpha'(t)|^2 + |\alpha'(t)|^2(\alpha'(t)\cdot\alpha''(t))^2,$$

which simplifies to

$$|\alpha'(t)|^2[|\alpha''(t)|^2|\alpha'(t)|^2 - (\alpha'(t)\cdot\alpha''(t))^2].$$

But

$$|\alpha''(t)|^2|\alpha'(t)|^2 - (\alpha'(t)\cdot\alpha''(t))^2 = |\alpha'(t)\wedge\alpha''(t)|^2,$$

where we have used the formula

$$(u\wedge v)\cdot(u\wedge v) = \begin{vmatrix} u\cdot u & u\cdot v\\ u\cdot v & v\cdot v\end{vmatrix}.$$

Thus

$$|\beta''(s)|^2 = \frac{|\alpha'(t)|^2|\alpha'(t)\wedge\alpha''(t)|^2}{|\alpha'(t)|^8},$$

so

$$|\beta''(s)| = \frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}.$$

This is the curvature of $\beta$ at $s$.

c. Let $b_\beta(s)$ be the binormal vector of $\beta$ at $s$, where $t = \phi^{-1}(s)$. Then

$$\begin{aligned}b_\beta(s) &= \beta'(s)\wedge\frac{\beta''(s)}{|\beta''(s)|}\\ &= \frac{\alpha'(t)\wedge\alpha''(t)}{|\alpha'(t)\wedge\alpha''(t)|}.\end{aligned}$$

Now write

$$\frac{d}{dt}\left(\frac{1}{|\alpha'(t)\wedge\alpha''(t)|}\right)=\frac{g(t)}{|\alpha'(t)\wedge\alpha''(t)|^3}$$

with

$$g(t)=-(\alpha'(t)\wedge\alpha''(t))\cdot(\alpha'(t)\wedge\alpha^{(3)}(t)).$$

Then

$$\begin{aligned}b'_\beta(s)&=\left[\frac{\alpha'(t)\wedge\alpha^{(3)}(t)}{|\alpha'(t)\wedge\alpha''(t)|}+\frac{g(t)(\alpha'(t)\wedge\alpha''(t))}{|\alpha'(t)\wedge\alpha''(t)|^3}\right]\frac{dt}{ds}\\&=\left[\frac{\alpha'(t)\wedge\alpha^{(3)}(t)}{|\alpha'(t)\wedge\alpha''(t)|}+\frac{g(t)(\alpha'(t)\wedge\alpha''(t))}{|\alpha'(t)\wedge\alpha''(t)|^3}\right]\frac{1}{|\alpha'(t)|}.\end{aligned}$$

Let $\tau_\beta(s)$ be the torsion of $\beta$ at $s$. Then

$$\begin{aligned}\tau_\beta(s)&=b'_\beta(s)\cdot\frac{\beta''(s)}{|\beta''(s)|}\\&=\frac{1}{|\beta''(s)||\alpha'(t)|}\left(\frac{(\alpha'(t)\wedge\alpha^{(3)}(t))\cdot\alpha''(t)}{|\alpha'(t)|^2|\alpha'(t)\wedge\alpha''(t)|}\right),\end{aligned}$$

since $\alpha'(t)\wedge\alpha^{(3)}(t)$ is orthogonal to $\alpha'(t)$, and $\alpha'(t)\wedge\alpha''(t)$ is orthogonal to both $\alpha'(t)$ and $\alpha''(t)$. Now

$$(\alpha'(t)\wedge\alpha^{(3)}(t))\cdot\alpha''(t)=-(\alpha'(t)\wedge\alpha''(t))\cdot\alpha^{(3)}(t)$$

(why?); we therefore obtain

$$\tau_\beta(s)=-\frac{(\alpha'(t)\wedge\alpha''(t))\cdot\alpha^{(3)}(t)}{|\alpha'(t)\wedge\alpha''(t)|^2}.$$

d. In terms of $x(t)$ and $y(t)$, we have that

$$\beta'(s)=\frac{(x'(t),y'(t))}{(x'(t)^2+y'(t)^2)^{\frac{1}{2}}},$$

so that the normal vector is given by

$$n(s)=(x'(t)^2+y'(t)^2)^{-\frac{1}{2}}(-y'(t),x'(t)).$$

But, from part b),

$$\begin{aligned}\beta''(s)&=\frac{(x''(x')^2+x''(y')^2-(x')^2x''-x'y'y'',y''(x')^2+y''(y')^2-y'x'x''-(y')^2y'')}{[(x')^2+(y')^2]^2}\\&=\frac{(x''(y')^2-x'y'y'',y''(x')^2-y'x'x'')}{[(x')^2+(y')^2]^2}\\&=[(x')^2+(y')^2]^{-\frac{3}{2}}(x'y''-x''y')n(s).\end{aligned}$$

This proves that the signed curvature of $\beta$ at $s$ is

$$(x'(t)^2+y'(t)^2)^{-\frac{3}{2}}(x'(t)y''(t)-x''(t)y'(t)).$$

1-5.14. We may assume that $\alpha$ is parametrized by arc length. At $t_0$, the function $f(t) = \alpha(t) \cdot \alpha(t)$ attains a maximum. Therefore, $f'(t_0) = 0$ and $f''(t_0) \leq 0$. Now

$$f'(t) = 2\alpha'(t) \cdot \alpha(t)$$

and

$$\begin{aligned} f''(t) &= 2\alpha''(t) \cdot \alpha(t) + 2\alpha'(t) \cdot \alpha'(t) \\ &= 2[(\alpha''(t) \cdot \alpha(t)) + 1], \end{aligned}$$

since $\alpha$ is parametrized by arc length. We then have

$$\alpha''(t_0) \cdot \alpha(t_0) + 1 \leq 0,$$

so

$$|\alpha''(t_0) \cdot \alpha(t_0)| \geq 1.$$

But, by the Cauchy-Schwarz inequality,

$$|\alpha''(t_0) \cdot \alpha(t_0)| \leq |\alpha''(t_0)||\alpha(t_0)| = |k(t_0)||\alpha(t_0)|,$$

so

$$|k(t_0)| \geq \frac{1}{|\alpha(t_0)|}.$$