NAME Solutions TA'S NAME ____________

STUDENT ID ____________ SECTION ____________

Math 124C
Winter 2012

Midterm 2
February 21, 2012

Point totals are indicated in parentheses. You must show your work to receive credit. Unless indicated otherwise, all answers must be exact.

(16) 1. Compute the derivative of the following functions. You need not simplify your answers.

a. $y = \sqrt{\ln(\sqrt{1-x})}$

b. $y = \left(\dfrac{x^2+1}{x+1}\right)^{\frac{4}{3}}$

c. $y = (\sin x)^x$

d. $y = \tan^{-1}(\sin^{-1}(x^2))$

a. $y = \sqrt{\ln(\sqrt{1-x})} = \sqrt{\frac{1}{2}\ln(1-x)}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1}{2}\ln(1-x)}} \cdot \frac{1}{2} \cdot \frac{-1}{1-x} = \frac{-1}{4(1-x)\sqrt{\frac{1}{2}\ln(1-x)}}$$

b. $$\frac{dy}{dx} = \frac{4}{3}\left(\frac{x^2+1}{x+1}\right)^{1/3}\left[\frac{2x(x+1)-(x^2+1)}{(x+1)^2}\right]$$

c. $\ln y = \ln[(\sin x)^x] = x\ln(\sin x)$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + \frac{x \cdot \cos x}{\sin x}$$

$$\frac{dy}{dx} = (\sin x)^x\left[\ln(\sin x) + \frac{x\cos x}{\sin x}\right]$$

d. $$\frac{dy}{dx} = \frac{1}{1+(\sin^{-1}(x^2))^2} \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x$$

(12) 2. Consider the curve $xy + e^{(y^2)} = e^{x+1}$.

(8) a. Find the equation of the tangent line to this curve at the point $(0, -1)$.

(4) b. Approximate the value of the x-coordinate of the point on the curve near $(0, -1)$ whose y-coordinate is -1.05. Give a decimal answer.

a. $y + x\frac{dy}{dx} + 2y\frac{dy}{dx}e^{y^2} = e^{x+1}$

When $x = 0$ and $y = -1$:

$$-1 - 2\frac{dy}{dx}e = e$$

$$\frac{dy}{dx} = \frac{e+1}{-2e} = -\frac{1}{2} - \frac{1}{2e}$$

equation of tangent line:

$$y + 1 = \left(-\frac{1}{2} - \frac{1}{2e}\right)x$$

$$y = \left(-\frac{1}{2} - \frac{1}{2e}\right)x - 1$$

b. Near $(0,-1)$, the curve is the graph of a function $y(x)$ with $y(0) = -1$. In a) we computed $y'(0) = -\frac{1}{2} - \frac{1}{2e}$.

By the tangent line approximation,

$$y(x) \approx y(0) + y'(0)x = -1 - \left(\frac{1}{2} + \frac{1}{2e}\right)x$$

for x near 0. We want to find x near 0 with $y(x) = -1.05$;

thus

$$-1.05 \approx -1 - \left(\frac{1}{2} + \frac{1}{2e}\right)x.$$

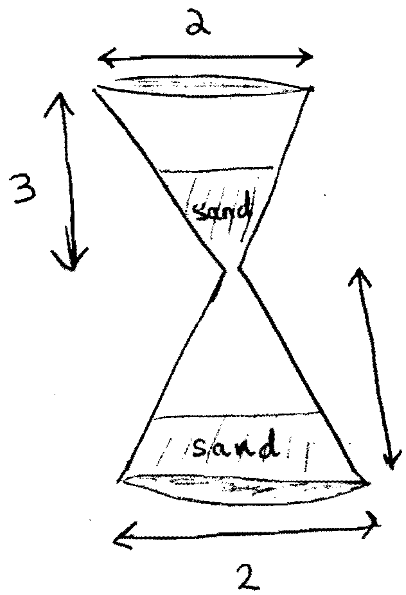
Solving for x yields

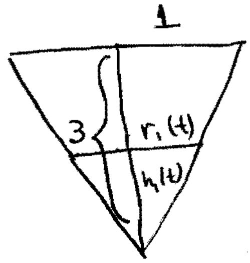
$$x \approx \frac{.05}{\frac{1}{2} + \frac{1}{2e}} = .0731$$

(14) 3. An hourglass consists of an inverted cone on top of another cone, with dimensions in inches as shown below. Sand in the top cone is dripping into the bottom cone. When the sand in the top cone has height 1.5 inches, the height of sand in the bottom cone is 1 inch, and the height of sand in the top cone is decreasing at the rate of .5 inch/minute.

a. What is the rate, in cubic inches per minute, that the sand in the top cone is drippping into the bottom cone, at this time?

b. What is the rate, in inches per minute, that the height of sand in the bottom cone is increasing at this time?





a. Let $V_1(t)$ = volume of sand (in cubic inches) at time t (in minutes)

$h_1(t)$ = height of sand (in inches) at time t

$$V_1(t) = \frac{1}{3}\pi r_1(t)^2 h_1(t)$$

By similar triangles,

$$\frac{r_1(t)}{1} = \frac{h_1(t)}{3}, \quad \text{so} \quad V_1(t) = \frac{\pi}{27} h_1(t)^3$$

Thus

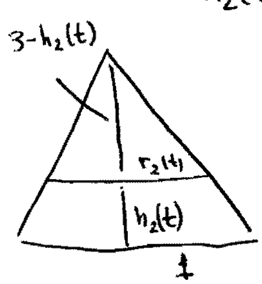
$$\frac{dV_1}{dt} = \frac{\pi}{9} h_1(t)^2 \frac{dh_1}{dt}.$$

When $h_1(t) = 3/2$, $\frac{dh_1}{dt} = -\frac{1}{2}$, so $\frac{dV_1}{dt} = -\frac{\pi}{8}$

Sand is dripping into the bottom cone at the rate of $\frac{\pi}{8}$ cubic inches per minute at this time.

b. Let $V_2(t)$ = volume of sand in bottom cone at time t

$h_2(t)$ = height of sand in bottom cone at time t



$$V_2(t) = \frac{1}{3}\pi \cdot 3 - \frac{1}{3}\pi r_2(t)^2 (3 - h_2(t))$$
$$= \pi - \frac{1}{27}\pi (3 - h_2(t))^3$$

(cont.)

Thus

$$\frac{dV_2}{dt} = -\frac{1}{9}\pi(3-h_2(t))^2\left(-\frac{dh_2}{dt}\right) = \frac{1}{9}\pi(3-h_2(t))^2\frac{dh_2}{dt}.$$

By a), $\frac{dV_2}{dt} = \frac{\pi}{8}$ when $h_2(t) = 1$, so at this time,

$$\frac{\pi}{8} = \frac{4}{9}\pi\frac{dh_2}{dt}$$

$$\frac{dh_2}{dt} = \frac{9}{32}$$ inches per minute.

(12) 4. A curve is given parametrically by the equations

$$\begin{aligned} x(t) &= 4t^3 + 3t^2 \\ y(t) &= 6t^2 + 6t. \end{aligned}$$

For what value of $t > 0$ does the tangent line to the curve at $(x(t), y(t))$ pass through the point $(0, 2)$?

At time t, the tangent line to the curve at $(x(t), y(t))$ has slope

$$\frac{y'(t)}{x'(t)} = \frac{12t+6}{12t^2+6t} = \frac{1}{t}$$

The equation of this tangent line is therefore

$$(y - 6t^2 - 6t) = \frac{1}{t}(x - 4t^3 - 3t^2).$$

For this line to pass through $(0,2)$ we must have

$$(2 - 6t^2 - 6t) = -\frac{1}{t}(4t^3 + 3t^2) = -4t^2 - 3t$$

$$2 - 2t^2 - 3t = 0$$

$$-(2t^2 + 3t - 2) = 0$$

The roots of $2t^2+3t-2$ are $\frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4} = \frac{1}{2}, -2.$

Answer: $t = \frac{1}{2}$.