NAME Solutions TA'S NAME ______________

STUDENT ID ______________ SECTION ______________

Math 124K
Fall 2012

Midterm 2
November 20, 2012

Point totals are indicated in parentheses. You must show your work to receive credit. Unless indicated otherwise, all answers must be exact.

(15) 1. Compute the derivative of the following functions. You need not simplify your answer, but your final answers must give the derivative in terms of x.

a. $y = [x\sin(x^3)]^{\sqrt{2}}$

b. $y = \tan^{-1}((x^2+1)^{\frac{1}{4}})$

c. $y = \ln\left[\left(\frac{1+x}{1+x^2}\right)^x\right]$

a. $\frac{dy}{dx} = \sqrt{2}\,[x\sin(x^3)]^{\sqrt{2}-1}\left(\sin(x^3) + x\cdot\cos(x^3)\cdot 3x^2\right)$

$= \sqrt{2}\,[x\sin(x^3)]^{\sqrt{2}-1}\left(\sin(x^3) + 3x^3\cos(x^3)\right)$

b. $\frac{dy}{dx} = \frac{1}{1+\sqrt{x^2+1}}\left(\frac{1}{4}(x^2+1)^{-3/4}\cdot 2x\right)$

$= \frac{1}{1+\sqrt{x^2+1}}\left(\frac{x}{2}(x^2+1)^{-3/4}\right)$

c. $y = x\ln(1+x) - x\ln(1+x^2)$

$\frac{dy}{dx} = \ln(1+x) + \frac{x}{1+x} - \ln(1+x^2) - \frac{2x^2}{1+x^2}$

(16) 2. Suppose that a particle is traveling along the curve $4\sqrt{xy+y}+xy^2-x^2=-13$.

(8) a. Find the equation of the tangent line to this curve at the point $(-3,-2)$.

(4) b. Suppose that the position of the particle at time t is given by $(x(t), y(t))$. Recall then that the speed $s(t)$ of the particle at time t is $s(t)=\sqrt{x'(t)^2+y'(t)^2}$. If the particle is at $(-3,-2)$ at time $t=0$ and its speed at this time is 1 with positive horizontal velocity, find $x'(0)$ and $y'(0)$.

(4) c. With the same assumptions as in part b), use the tangent line approximation to estimate the position of the particle at time $t=0.1$.

a. Implicitly differentiate:

$$\frac{2\cdot\left(y+x\frac{dy}{dx}+\frac{dy}{dx}\right)}{\sqrt{xy+y}} + y^2 + 2xy\frac{dy}{dx} - 2x = 0$$

Plug in $x=-3$, $y=-2$ to compute $\frac{dy}{dx}$ when $x=-3$, $y=-2$:

$$-2+(-3)\frac{dy}{dx}+\frac{dy}{dx}+4+12\frac{dy}{dx}+6=0$$

$$10\frac{dy}{dx}=-8$$

$$\frac{dy}{dx}=-4/5$$

equation of tangent line: $y+2=-\frac{4}{5}(x+3)$

$$y=-\frac{4}{5}x-22/5$$

b. $\frac{y'(0)}{x'(0)}=\frac{dy}{dx}=-4/5$ so $y'(0)=-\frac{4}{5}x'(0)$.

Therefore,

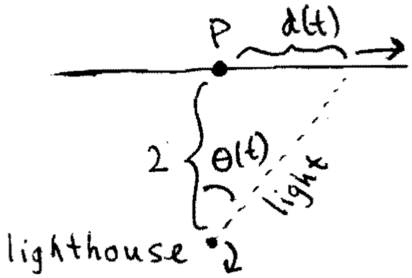
$$1=\sqrt{x'(0)^2+\frac{16}{25}x'(0)^2}=x'(0)\frac{\sqrt{41}}{5}$$

$$x'(0)=\frac{5}{\sqrt{41}},\quad y'(0)=-\frac{4}{5}\cdot\frac{5}{\sqrt{41}}=\frac{-4}{\sqrt{41}}$$

c. $x(0.1)\approx x(0)+x'(0)(0.1-0)=-3+\frac{5}{10\sqrt{41}}=-3+\frac{1}{2\sqrt{41}}$

$y(0.1)\approx y(0)+y'(0)(0.1-0)=-2-\frac{4}{10\sqrt{41}}=-2-\frac{2}{5\sqrt{41}}$

(12) 3. A lighthouse is located on a small island 2 kilometers away from the nearest point P on a straight shoreline, and its light makes 6 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 kilometer from P? (Remember to give an exact, not a decimal, answer.)



Let $d(t)$ = distance (in km) of beam from P at time t (in min)

$\theta(t)$ = pictured angle at time t.

We know: $\theta'(t) = 12\pi$

Want to find: $d'(t)$ at time t when $d(t) = 1$.

At any time t:

$$\tan\theta(t) = \frac{d(t)}{2}$$

Differentiate both sides with respect to t:

$$\sec^2\theta(t)\cdot\theta'(t) = \frac{d'(t)}{2}$$

When $d(t)=1$, $\tan\theta(t) = \frac{1}{2}$, so

$$\sec^2\theta(t) = \tan^2\theta(t) + 1 = 5/4$$

Therefore, when $d(t) = 1$, we have

$$12\pi\cdot 5/4 = \frac{d'(t)}{2}$$

$$d'(t) = 30\pi \text{ km/min.}$$

(12) 4. Let b be a positive constant, and consider the curve C given by the parametric equations

$$\begin{aligned} x(t) &= t^b\cos(\pi t) \\ y(t) &= t^b\sin(\pi t) \end{aligned}$$

for t in $(0,3)$.

(8) a. Find the slope of the tangent line to the curve at time $t > 0$.

(4) b. For what value of b will the tangent line to the curve at $(-1,0)$ be $y = 3x+3$?

a. $x'(t) = bt^{b-1}\cos(\pi t) - \pi t^b \sin \pi t$

$y'(t) = bt^{b-1}\sin(\pi t) + \pi t^b \cos \pi t$

Slope of tangent line at time t $= \dfrac{y'(t)}{x'(t)} = \dfrac{bt^{b-1}\sin(\pi t) + \pi t^b\cos \pi t}{bt^{b-1}\cos(\pi t) - \pi t^b \sin \pi t}$

b. $(-1,0) = (x(1), y(1))$, so the slope of the tangent line to the curve at this point is $\dfrac{y'(1)}{x'(1)} = \dfrac{-\pi}{-b}$. We want $\frac{-\pi}{-b} = 3$; thus

$b = \frac{\pi}{3}$.