Identifiability and the foundations of sensitivity analysis

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Goals

- Identifiable sets are central for algorithms and sensitivity analysis.
- Existence, calculus, properties.
- Connection to critical cones (Generalized Reduction Lemma).
- Illustration: Spectral functions.
- Generic existence (semi-algebraic setting).

Motivation (Algorithms)

Many algorithms for minimizing $f : \mathbf{R}^n \to \overline{\mathbf{R}}$,

- Subgradient (Gradient) projection methods,
- Newton-like methods,
- Proximal Point Algorithms,

produce iterates $x_k \to \bar{x}$, along with criticality certificates:

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- Foreshadowing: these problems are equivalent in a much stronger sense!

Finite identification

Definition (Identifiable sets)

A set $\mathcal{M} \subset \mathbf{R}^n$ is identifiable at $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ if

$$\left.\begin{array}{c} x_i \to \bar{x}, v_i \to \bar{v} \\ v_i \in \partial f(x_i) \end{array}\right\} \Longrightarrow x_i \in \mathcal{M} \text{ for all large } i,$$

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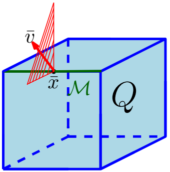
Let $f(x) = \delta_Q(x) - \langle \bar{v}, x \rangle$.



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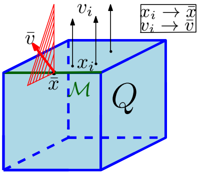

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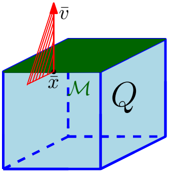

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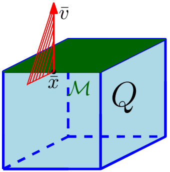
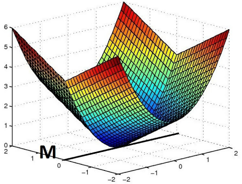


In this case $\mathcal{M} = \bar{x} + K_Q(\bar{x}, \bar{v})$.

Example

The "nicest" situation:



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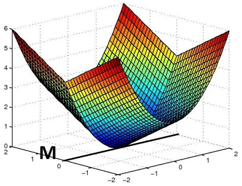

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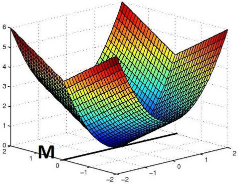

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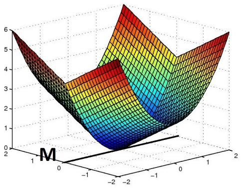

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- "Finite identification" considered implicitly by a number of authors: Bertsekas '76, Rockafellar '76, Calamai '87, Burke and Moré '88, Dunn '87, Ferris '91, Wright '93, Lewis and Hare '07...

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- $\bar{x}$ *is a (strict) local minimizer of* $f \iff \bar{x}$ *is a (strict) local minimizer of* $f\big|_{\mathcal{M}}$.
- f *grows quadratically near* $\bar{x} \iff f\big|_{\mathcal{M}}$ *grows quadratically near* $\bar{x}$.
- f *is tilt-stable at* $\bar{x} \iff f\big|_{\mathcal{M}}$ *is tilt-stable at* $\bar{x}$.

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Locally minimal identifiable sets may fail to exist in general (e.g. $f(x, y) = \sqrt{x^4 + y^2}$).

Existence and calculus

Locally minimal identifiable sets exist for

- fully amenable functions: $f(x) = g(F(x))$ where
  1. F is $\mathbf{C}^2$-smooth,
  2. g is (convex) piecewise quadratic,
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A strong chain rule is available for composite functions

$$f(x) = g(F(x)).$$

Dimension Reduction

The critical cone of a convex set Q at $\bar{x}$ for $\bar{v} \in N_Q(\bar{x})$ is

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Let $\mathcal{M}$ be a (prox-regular) identifiable set at $(\bar{x},\bar{v}) \in \operatorname{gph} N_Q(\bar{x})$. Then

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May use this to study nonpolyhedral variational inequalities!

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Proposition (D-Lewis)

Let $\mathcal{M}$ be a $\mathbf{C}^2$-manifold. Then

$$\mathcal{M} \text{ is identifiable at } (\bar{x}, \bar{v})$$

if and only if

- *$\mathcal{M}$ is a partly smooth manifold at $\bar{x}$ (for $\bar{v}$),*
- *$\bar{v} \in \operatorname{ri} \partial f(\bar{x})$,*
- *f is prox-regular at $\bar{x}$ for $\bar{v}$.*

Lifts of identifiable manifolds

Consider $\mathbf{S}^n := \{n \times n \text{ symmetric matrices}\}$ and the eigenvalue map

$$A \mapsto (\lambda_1(A), \ldots, \lambda_n(A)),$$

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Identifiable manifolds "lift": (D, Lewis), (Daniilidis, Malick, Sendov)

$$\mathcal{M} \text{ identifiable manifold at } (\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$$
$$\Longrightarrow \lambda^{-1}(\mathcal{M}) \text{ identifiable manifold at } (\bar{X}, \bar{V}) \in \operatorname{gph} \partial(f \circ \lambda).$$

Generic Properties

History: Rockafellar-Spingarn '79, considered problems

$$\begin{aligned} P(v,u): \quad & \min \quad f(x) - \langle v, x\rangle, \\ & \text{s.t.} \quad g_i(x) \le u_i, \ \text{for all } i \in I := \{1, \ldots, m\}, \end{aligned}$$

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Theorem (Rockafellar-Spingarn '79)

- *For almost all (v,u), at every minimizer of $P(v,u)$:*

Active manifold: active gradients are independent

Strict complementarity: multipliers are strictly positive and

Quadratic growth: objective function grows quadratically.

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[Convex semi-algebraic case considered in Bolte, Daniilidis, Lewis '11].

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Theorem (Ioffe)

For semi-algebraic $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, for generic $v \in \mathbf{R}^m$, have

$$x \in F^{-1}(v) \Longrightarrow F \textit{ is metrically regular at } (x, v).$$

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Theorem (Ioffe)

For semi-algebraic $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, *for generic* $v \in \mathbf{R}^m$, *have*

$$x \in F^{-1}(v) \Longrightarrow F \text{ is metrically regular at } (x, v).$$

Further, if $\dim \operatorname{gph} F = n = m$, then strong metric regularity is typical.

Generic Properties

Semi-algebraic subdifferential graphs are not too big, not too small, but just right:

Theorem (D, Ioffe, Lewis)

For lsc, semi-algebraic $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$, we have

$$\dim \operatorname{gph} \partial f = n,$$

even locally around any pair $(x, v) \in \operatorname{gph} \partial f$.

Metric regularity is typical (Nonsmooth Sard's theorem):

Theorem (Ioffe)

For semi-algebraic $F \colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$, for generic $v \in \mathbf{R}^m$, have

$$x \in F^{-1}(v) \Longrightarrow F \text{ is metrically regular at } (x, v).$$

Further, if $\dim \operatorname{gph} F = n = m$, then strong metric regularity is typical.
Strong metric regularity of ∂f (i.e. tilt-stability) is equivalent to a uniform quadratic growth condition (D, Lewis '12).

Summary

- Presented the intuitive notion of identifiable sets.
- Showed how identifiable sets capture the essence of previously developed concepts (dimension reduction, critical cones, optimality conditions).
- Illustration: spectral functions.
- Generic properties of semi-algebraic optimization problems.

References

- **Optimality, identifiability, and sensitivity**, D-Lewis, submitted to Math. Programming Ser. A.
- **The dimension of semi-algebraic subdifferential graphs**, D-Ioffe-Lewis. Nonlinear Analysis: Theory, methods, and applications, 75(3), 1231-1245, 2012.
- **Semi-algebraic functions have small subdifferentials**, D-Lewis. to appear in Math. Programming Ser. B.

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Advertisement: **Tame variational analysis, a survey**, D-Ioffe-Lewis, to appear (at some point).

Thank you.