Robust stochastic optimization with the proximal point method

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Abstract

Standard results in stochastic convex optimization bound the number of samples that an algorithm needs to generate a point with small function value in expectation. In this work, we show that a wide class of such algorithms on strongly convex problems can be augmented with sub-exponential confidence bounds at an overhead cost that is only polylogarithmic in the condition number and the confidence level. We discuss consequences both for streaming and offline algorithms.

1 Introduction

Stochastic convex optimization lies at the core of modern statistical and machine learning. Standard results in the subject bound the number of samples that an algorithm needs to generate a point with small function value in expectation. More nuanced high probability guarantees for numerical methods are rarer, and typically either rely on “light-tails” assumptions or exhibit worse sample complexity. To address this issue, we show that a wide class of stochastic algorithms for strongly convex problems can be augmented with sub-exponential confidence bounds at an overhead cost that is only polylogarithmic in the condition number and logarithmic in the confidence level. The procedure we propose, called proxBoost, is elementary and combines two well-known ingredients: robust distance estimation and the proximal point method.

To illustrate the proposed procedure, consider the optimization problem

$$\min_x f(x)$$

where $f: \mathbb{R}^d \to \mathbb{R}$ is a $\mu$-strongly convex function with $L$-Lipschitz continuous gradient. We will later consider the more general class of convex composite problems. We aim to develop generic procedures that equip stochastic algorithms with high confidence guarantees. Consequently, it will be convenient to treat such algorithms as black boxes. More formally, suppose that the function $f$ may only be accessed through a minimization oracle $\mathcal{M}_f(\cdot)$, which on input $\epsilon > 0$, returns a point $x_\epsilon$ satisfying the low confidence bound

$$\mathbb{P}(f(x_\epsilon) - \min f \leq \epsilon) \geq \frac{2}{3}. \quad (1.1)$$
By Markov’s inequality, minimization oracles arise from any algorithm that can generate a point $x$ satisfying $\mathbb{E}f(x) - \min f \leq \epsilon/3$. For example, oracles for minimizing an expectation $f(x) = \mathbb{E}_z [f(x, z)]$ may be constructed from streaming algorithms or from offline empirical risk minimization methods.

The procedure introduced in this paper executes a minimization oracle multiple times in order to boost its confidence. To quantify this overhead, let $\mathcal{C}_M(\epsilon, f)$ denote the cost of the oracle call $M_f(\epsilon)$. It is natural to assume that the cost is decreasing in $\epsilon$ and increasing in the condition number $\tau := L/\mu$. The cost may also depend on other parameters, such as initialization quality and bounds on optimal value, but we ignore these for a moment. Given a minimization oracle and its cost, we investigate the following question:

Is there a procedure within this oracle model of computation that returns a point $x$ satisfying the high confidence bound

$$\mathbb{P}(f(x) - \min f \leq \epsilon) \geq 1 - p$$

at a total cost that is only a “small” multiple of $\mathcal{C}_M(\epsilon, f) \cdot \ln(1/p)$?

We will see that the answer is yes, with the total cost on the order of

$$\log \left( \frac{\log(\tau)}{p} \right) \log(\tau) \cdot \mathcal{C}_M \left( \frac{\epsilon}{\log(\tau)}, f \right).$$

Thus, high probability bounds are achieved with a small cost increase, which depends only logarithmically on $1/p$ and polylogarithmically on the condition number $\tau$.

Before introducing our approach, we discuss two techniques for boosting the confidence of a minimization oracle, both of which have limitations. As a first approach, one may query the oracle $M_f(\epsilon)$ multiple times and pick the “best” iterate from the batch. This approach is flawed since often one cannot test which iterate is “best” without increasing sample complexity. To illustrate, consider estimating the expectation $f(x) = \mathbb{E}_z [f(x, z)]$ to $\epsilon$-accuracy for a fixed point $x$. This task amounts to approximate mean estimation, which may require on the order of $1/\epsilon^2$ samples, even under sub-Gaussian assumptions [8].

In this paper, the cost $\mathcal{C}_M(\epsilon, f)$ will typically scale at worst as $1/\epsilon$, and therefore mean estimation would significantly degrade the overall sample complexity.

As the second approach, strong convexity immediately implies the distance estimate

$$\mathbb{P}(\|x - \bar{x}\| \leq \sqrt{2\epsilon/\mu}) \geq \frac{2}{3},$$

where $\bar{x}$ is the minimizer of $f$. Given this bound, one may apply the robust distance estimation technique of [35, p. 243] and [19] to choose a point near $\bar{x}$: Run $m$ trials of $M_f(\epsilon)$ and find one iterate $x_m$ around which the other points “cluster”. Then the point $x_m$ will be within a distance of $3\sqrt{2\epsilon/\mu}$ from $\bar{x}$ with probability $1 - \exp(-m/18)$. The downside of this strategy is that when converting naively back to function values, the suboptimality gap becomes $f(x_m) - \min f \leq \frac{\tau}{2}\|x_m - \bar{x}\|^2 \leq 9\tau\epsilon$. Thus the function gap at $x_m$ may be significantly larger than the expected function gap at $x$, by a factor of

2
the condition number. Therefore, robust distance estimation exhibits a trade-off between robustness and efficiency.

The trade-off between robustness and efficiency disappears for perfectly conditioned losses. Therefore, it appears plausible that one might avoid the $\tau$ factor through an iterative algorithm that begins with a low accuracy solution to the original problem and then iteratively finds higher accuracy solutions to nearby, better conditioned problems. This is the strategy we explore here. The proxBoost procedure embeds the robust distance estimation technique inside a proximal point method. The algorithm begins by declaring the initial point $x_0$ to be the output of the robust distance estimator on $f$. Then the better conditioned function

$$f^t(x) := f(x) + \frac{\mu t^2}{2} ||x - x_t||^2,$$

is formed and the next iterate $x_{t+1}$ is declared to be the output of the robust distance estimator on $f^t$. The procedure is effective since the conditioning of $f^t$ rapidly improves with $t$, which makes the robust distance estimator more efficient as the counter $t$ grows.

The proxBoost procedure can be applied to a wide class of stochastic minimization oracles, for example, streaming or empirical risk minimization (ERM) algorithms. For these problems, the loss $f$ takes the form

$$f(x) = \mathbb{E}_{z \sim P}[f(x, z)],$$

where the population data $z$ follows a fixed unknown distribution $P$ and the loss $f(\cdot, z)$ is convex and smooth for a.e. $z \in P$. The cost of streaming or ERM oracles is then measured by the number of samples drawn from $P$. We now illustrate the consequences of proxBoost for these oracles.

1.1 Streaming Oracles

Stochastic gradient methods can be treated as minimization oracles $M_f(\epsilon)$ with cost $C_M(\epsilon, f)$ that is measured by the number stochastic gradient estimates needed to reach functional accuracy $\epsilon$ in expectation. An algorithm with minimal such cost was proposed by Ghadimi and Lan [16]. It generates a point $x_\epsilon$ satisfying $\mathbb{E}[f(x_\epsilon) - \min f] \leq \epsilon$ with

$$O\left(\sqrt{\tau \ln \left(\frac{\Delta_{in}}{\epsilon}\right)} + \frac{\sigma^2}{\mu \epsilon}\right),$$

stochastic gradient evaluations, where the quantity $\sigma^2$ is an upper bound on the variance of the stochastic gradient estimator $\nabla f(x, z)$ and $\Delta_{in}$ is a known upper bound on the initial function gap $\Delta_{in} \geq f(x_0) - f^*$. A simpler algorithm with a similar efficiency estimate was recently presented by Kulunchakov and Mairal [24], and was based on estimate sequences. Aybat et al. [4] present an algorithm with similar efficiency, but in contrast to previous work, it does not require the variance $\sigma^2$ and the initial gap $\Delta_{in}$ as inputs.

It is intriguing to ask if one can equip the stochastic gradient method and its accelerated variant with high confidence guarantees. In their original work [15,16], Ghadimi and
Lan provide an affirmative answer under the additional assumption that the stochastic gradient estimator has light tails. The very recent preprint of Juditsky-Nazin-Nemirovsky-Tsybakov [22] shows that one can avoid the light tail assumption for the basic stochastic gradient method, and for mirror descent more generally, by truncating the gradient estimators. High confidence bounds for the accelerated method, without light tail assumptions, remain open.

In this work, the optimal method of [16] will be used as a minimization oracle within proxBoost, allowing us to nearly match the efficiency estimate (1.4) without “light-tail” assumptions. Equipped with this oracle, proxBoost returns a point \( x \) satisfying

\[
\mathbb{P}[f(x) - f^* \leq \epsilon] \geq 1 - p,
\]

and the overall cost of the procedure is

\[
\tilde{O}\left( \log \left( \frac{1}{p} \right) \left( \sqrt{T} \ln \left( \frac{\Delta_m}{\epsilon} \vee \tau \right) + \frac{\sigma^2}{\mu \epsilon} \right) \right).
\]

Here, \( \tilde{O}(\cdot) \) only suppresses logarithmic dependencies in \( \tau \); see Section 5 for details. Thus for small \( \epsilon \), the sample complexity of the robust procedure is roughly \( \log(1/p) \) times the efficiency estimate (1.4) of the low-confidence algorithm. In this paper, we also provide similar accelerated guarantees for additive convex composite problems, by using the routine of [22] in the last step of proxBoost.

### 1.2 Empirical Risk Minimization Oracles

An alternative approach to streaming algorithms, such as the stochastic gradient method, is based on empirical risk minimization (ERM). Namely, we may draw i.i.d. samples \( z_1, \ldots, z_n \sim \mathbb{P} \) and minimize the empirical average

\[
\min_x f_S(x) := \frac{1}{n} \sum_{i=1}^{n} f(x, z_i). \tag{1.5}
\]

A key question is to determine the number \( n \) of samples that would ensure that the minimizer \( x_S \) of the empirical risk \( f_S \) has low generalization error \( f(x_S) - \min f \), with reasonably high probability. There is a vast literature on this subject; see for example [5,19,38,39]. We build here on the work of Hsu-Sabato [19], who focused on high confidence guarantees for nonnegative losses \( f(x, z) \). They showed that the empirical risk minimizer \( x_S \) yields a robust distance estimator of the true minimizer of \( f \), by the aforementioned resampling technique. As a consequence they deduced that the ERM learning rule can find a point \( x_S \) satisfying the relative error guarantee

\[
\mathbb{P}[f(x_S) \leq (1 + \gamma)f^*] \geq 1 - p,
\]

with the sample complexity \( n \) on the order of

\[
\mathcal{O}\left( \log \left( \frac{1}{p} \right) \frac{\hat{\tau} \gamma}{\gamma} \right).
\]
Loosely speaking, here $\tau$ and $\hat{\tau}$ are the condition numbers of $f$ and $f_S$, respectively. By embedding empirical risk minimization within \texttt{proxBoost}, we obtain an algorithm with the much better sample complexity

$$\tilde{O}\left(\log \left(\frac{1}{p}\right) \cdot \left(\frac{\hat{\tau}}{\gamma} + \hat{\tau}\right)\right),$$

where the symbol $\tilde{O}$ only suppresses polylogarithmic dependence on $\tau$ and $\hat{\tau}$.

\section*{Related literature}

Our paper rests on two pillars: the proximal point method and robust distance estimation. Both techniques have been well studied in the optimization and statistics literature. The proximal point method was introduced by Martinet \cite{30,31} and further popularized by Rockafellar \cite{37}. The construction is also closely related to the smoothing function of Moreau \cite{33}. Recently, there has been a renewed interest in the proximal point method, most notably due to its uses in accelerating variance reduced methods for minimizing finite sums of convex functions \cite{13,26,27,40}. The proximal point method has also featured prominently as a guiding principle in nonconvex optimization, with the works of \cite{2,8,10,12}. The stepsize schedule we use within the proximal point method is geometrically decaying, in contrast to the more conventional polynomially decaying schemes. Geometrically decaying schedules for subgradient methods were first used by Goffin \cite{18} and have regained some attention recently due to their close connection to the popular step-decay schedule in stochastic optimization \cite{4,14,41,42}.

Robust distance estimation has a long history. The estimator we use was first introduced in \cite{35}, p. 243, and can be viewed as a multivariate generalization of the median of means estimator \cite{1,20}. Robust distance estimation was further investigated in \cite{19} with a focus on high probability guarantees for empirical risk minimization. A different generalization based on the geometric median was studied in \cite{32}. Other recent articles related to the subject include median of means tournaments \cite{28}, robust multivariate mean estimators \cite{21,29}, and bandits with heavy tails \cite{7}.

One of the main applications of our techniques is to streaming algorithms. Most currently available results that establish high confidence convergence guarantees make sub-Gaussian assumptions on the stochastic gradient estimator \cite{15,17,23,34}. More recently there has been renewed interest in obtaining robust guarantees without the light-tails assumption. For example, the two works \cite{9,43} make use of the geometric median of means technique to robustly estimate the gradient in distributed optimization. A different technique was recently developed by Juditsky et al. \cite{22}, where the authors establish high confidence guarantees for mirror descent type algorithms by truncating the gradient.

The outline of the paper is as follows. Section \ref{sec:setting} presents the problem setting. Section \ref{sec:method} develops the \texttt{proxBoost} procedure. Section \ref{sec:empirical} presents consequences for empirical risk minimization, while Section \ref{sec:streaming} discusses consequences for streaming algorithms.
Problem setting

Throughout, we follow standard notation of convex optimization, as set out for example in the monographs [6,36]. We let $\mathbb{R}^d$ denote an Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. The symbol $B_\epsilon(x)$ will stand for the closed ball around $x$ of radius $\epsilon > 0$. We will use the shorthand interval notation $[1, m] := \{1, \ldots, m\}$ for any number $m \in \mathbb{N}$.

Consider a function $f: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. The effective domain of $f$, denoted $\text{dom} \, f$, consists of all points where $f$ is finite. The function $f$ is called $\mu$-strongly convex if the perturbed function $f - \frac{\mu}{2} \|\cdot\|^2$ is convex. We say that $f$ is $L$-smooth if it differentiable with $L$-Lipschitz continuous gradient. If $f$ is both $\mu$-strongly convex and $L$-smooth, then the two sided bound holds:

$$\frac{\mu}{2} \|x - \bar{x}\|^2 \leq f(x) - f(\bar{x}) \leq \frac{L}{2} \|x - \bar{x}\|^2 \quad \text{for all } x,$$

where $\bar{x}$ is the minimizer of $f$. We then define the condition number of $f$ to be $\tau := L/\mu$.

**Assumption 2.1.** Throughout this work, we consider the optimization problem

$$\min_{x \in \mathbb{R}^d} f(x)$$

where the function $f: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is $\mu$-strongly convex. We denote the minimizer of $f$ by $\bar{x}$ and its minimal value by $f^* := \min f$.

Let us suppose for the moment that the only access to $f$ is by querying a black-box procedure that estimates $\bar{x}$. Namely following [19] we will call a procedure $D(\epsilon)$ a weak distance oracle for the problem (2.2) if it returns a point $x$ satisfying

$$\mathbb{P}[\|x - \bar{x}\| \leq \epsilon] \geq \frac{2}{3}.$$  

(2.3)

We will moreover assume that when querying $D(\epsilon)$ multiple times, the returned vectors are all statistically independent. Weak distance oracles arise naturally in stochastic optimization both in streaming and offline settings. We will discuss specific examples in Sections 4 and 5. The numerical value $2/3$ plays no real significance and can be replaced by any fraction greater than a half.

It is well known from [35, p. 243] and [19] that the low-confidence estimate (2.3) can be improved to a high confidence guarantee by a resampling trick. Following [19], we define the robust distance estimator $D(\epsilon, m)$ to be the following procedure

**Algorithm 1:** Robust Distance Estimation $D(\epsilon, m)$

<table>
<thead>
<tr>
<th><strong>Input:</strong></th>
<th>trial count $m$, access to a weak distance oracle $D(\epsilon)$</th>
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</table>
| **Step**  | $i = 1, \ldots, m$:
|           | Compute $r_i = \min\{r \geq 0 : |B_r(y_i) \cap Y| > \frac{m}{2}\}$. |
|           | Set $i^* = \arg\min_{i \in [1, m]} r_i$.
| **Return**| $y_{i^*}$                                                   |
Thus the robust distance estimator $D(\epsilon, m)$ first generates $m$ statistically independent random points $y_1, \ldots, y_m$ by querying $m$ times the weak distance oracle $D(\epsilon)$. Then the procedure computes the smallest radius ball around each point $y_i$ that contains more than half of the generated points $\{y_1, \ldots, y_m\}$. Finally, the point $y_i^*$ corresponding to the smallest such ball is returned. The intuition behind this procedure is that by Chernoff’s bound, with high probability, the ball $B_\epsilon(\bar{x})$ will contain at least $m/2$ of the generated points. Therefore in this event, the estimate $r_i^* < 2\epsilon$ holds. Moreover since the two sets, $B_\epsilon(\bar{x})$ and $B_{r_i^*}(y_i^*)$ intersect, it follows that $\bar{x}$ and $y_i^*$ are within a distance of $3\epsilon$ of each other. For a complete argument, see [19, Propositions 8,9].

**Lemma 2.2** (Robust Distance Estimator). The point $x$ returned by $D(\epsilon, m)$ satisfies

$$\mathbb{P}[\|x - \bar{x}\| \leq 3\epsilon] \geq 1 - \exp\left(-\frac{m}{18}\right).$$

We seek to understand how one may use a robust distance estimator $D(\epsilon, m)$ to compute a point $x$ satisfying $f(x) - \min f \leq \delta$ with high probability, where $\delta > 0$ is a specified accuracy. As motivation, consider the case when $f$ is $L$-smooth. Then one immediate approach is to appeal to the upper bound in (2.1). Hence the point $x = D(\epsilon, m)$, with $\epsilon = \sqrt{\frac{2\delta}{9L}}$, satisfies the guarantee

$$\mathbb{P}\left(f(x) - f^* \leq \delta\right) \geq 1 - \exp\left(-\frac{m}{18}\right).$$

We will follow an alternative approach, which in concrete circumstances can significantly decrease the overall cost. The optimistic goal is to replace the accuracy $\epsilon \approx \sqrt{\frac{2\delta}{L}}$ used in the call to $D(\epsilon, m)$ by the potentially much larger quantity $\sqrt{\frac{\delta}{\mu}}$. The strategy we propose will apply a robust distance estimator $D$ to a sequence of optimization problems that are better and better conditioned, thereby amortizing the overall cost. In the initial step, we will simply apply $D$ to $f$ with the low accuracy $\sqrt{\frac{\delta}{\mu}}$. In step $i$, we will apply $D$ to a new function $f_i$, which has condition number $\tau_i \approx \frac{L + \mu^2}{\mu + \mu 2^i}$, with accuracy $\epsilon_i \approx \sqrt{\frac{\delta}{\mu + \lambda_i}}$.

Continuing this process for $T \approx \log_2(\tau)$ rounds, we arrive at accuracy $\epsilon_T \approx \sqrt{\frac{\delta}{\mu + L}}$ and a function $f_T$ that is nearly perfectly conditioned with $\tau_T \leq 2$. In this way, the total cost is amortized over the sequence of optimization problems. The key of course is to control the error incurred by varying the optimization problems along the iterations.

### 3 Main result

The procedure outlined at the end of the previous section can be succinctly described within the framework of an inexact proximal point method. Henceforth fix an increasing sequence of penalties $\lambda_0, \ldots, \lambda_T$ and a sequence of centers $x_0, \ldots, x_T$. For each index $i = 0, \ldots, T$, define the quadratically perturbed functions and their minimizers:

$$f^i(x) := f(x) + \frac{\lambda_i}{2}\|x - x_i\|^2, \quad \bar{x}_{i+1} := \arg\min_x f^i(x).$$
The exact proximal point method \cite{30,31,37} proceeds by inductively declaring \(x_i = \bar{x}_i\) for \(i \geq 1\). Since computing \(\bar{x}_i\) is in general impossible, we will instead monitor the error \(\|\bar{x}_i - x_i\|\). The following elementary result will form the basis for the rest of the paper. To simplify notation, we will set \(\bar{x}_0 := \arg\min f\) and \(\lambda_{-1} := 0\), throughout.

**Theorem 3.1** (Inexact proximal point method). For all \(j \geq 0\), the estimates hold:

\[
f^j(\bar{x}_{j+1}) - f^* \leq \sum_{i=0}^{j} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2, \tag{3.1}
\]

Consequently, we have the error decomposition:

\[
f(x_{j+1}) - \min f \leq (f^j(x_{j+1}) - f^j(\bar{x}_{j+1})) + \sum_{i=0}^{j} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2. \tag{3.2}
\]

Moreover, if \(f\) is \(L\)-smooth, then for all \(j \geq 0\) the estimate holds:

\[
f(x_j) - f^* \leq \frac{L + \lambda_j}{2} \|\bar{x}_j - x_j\|^2 + \sum_{i=0}^{j-1} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2. \tag{3.3}
\]

**Proof.** We first establish (3.1) by induction. To see the base case \(j = 0\), observe

\[f^0(\bar{x}_1) \leq f^0(\bar{x}_0) = f^* + \frac{\lambda_0}{2} \|\bar{x}_0 - x_0\|^2. \]

As the inductive assumption, suppose the estimate (3.1) holds up to iteration \(j - 1\). We then conclude

\[
f^j(\bar{x}_{j+1}) \leq f^j(\bar{x}_j) + \frac{\lambda_j}{2} \|\bar{x}_j - x_j\|^2 \\
\leq f^{j-1}(\bar{x}_j) + \frac{\lambda_j}{2} \|\bar{x}_j - x_j\|^2 \leq f^* + \sum_{i=0}^{j-1} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2,
\]

where the last inequality follows by the inductive assumption. This completes the proof of (3.1). To see (3.2), we observe using (3.1) the estimate

\[
f(x_{j+1}) - f^* \leq f^j(x_{j+1}) - f^* = (f^j(x_{j+1}) - f^j(\bar{x}_{j+1})) + f^j(\bar{x}_{j+1}) - f^* \\
\leq (f^j(x_{j+1}) - f^j(\bar{x}_{j+1})) + \sum_{i=0}^{j} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2.
\]

Inequality (3.3) for index \(j = 0\) follows from smoothness, while the general case \(j \geq 1\) follows from using the bound \(f^j(x_{j+1}) - f^j(\bar{x}_{j+1}) \leq \frac{L + \lambda_j}{2} \|\bar{x}_{j+1} - x_{j+1}\|^2\) in (3.2). \(\square\)
The main conclusion of Theorem 3.1 is the decomposition of the functional error described in (3.2). Namely, the estimate (3.2) upper bounds the error \( f(x_{j+1}) - \min f \) as the sum of the suboptimality in the last step \( f^T(x_{T+1}) - f^T(\bar{x}_{T+1}) \) and the errors \( \frac{\lambda_i}{2} \| x_i - x_i \|^2 \) incurred along the way. By choosing \( T \) sufficiently large, we can be sure that the function \( f^T \) is well-conditioned. Moreover in order to ensure that each term in the sum \( \frac{\lambda_i}{2} \| x_i - x_i \|^2 \) is of order \( \delta \), it suffices to guarantee \( \| x_i - x_i \| \leq \sqrt{\frac{2\delta}{\lambda_i}} \) for each index \( i \). Since \( \lambda_i \) is an increasing sequence, it follows that we may gradually decrease the tolerance on the errors \( \| x_i - x_i \| \), all the while improving the conditioning of the functions we encounter.

With this intuition in mind, we introduce the \texttt{proxBoost} procedure (Algorithm 2).

\begin{algorithm}
\caption{\texttt{proxBoost}(\( \delta, p, T \))}
\begin{algorithmic}
\Statex \textbf{Input:} \( \delta \geq 0 \), \( p \in (0, 1) \), \( T \in \mathbb{N} \)
\Statex Set \( \lambda_{-1} = 0 \), \( \epsilon_{-1} = \sqrt{\frac{2\delta}{\mu}} \)
\Statex Generate a point \( x_0 \) satisfying \( \| x_0 - \bar{x}_0 \| \leq \epsilon_{-1} \) with probability \( 1 - p \).
\For {\( j = 0, \ldots, T - 1 \)}
\Statex Set \( \epsilon_j = \sqrt{\frac{2\delta}{\mu+\lambda_j}} \)
\Statex Generate a point \( x_{j+1} \) satisfying
\Statex \[ \mathbb{P} \left[ \| x_{j+1} - \bar{x}_{j+1} \| \leq \epsilon_j \mid E_j \right] \geq 1 - p, \]
\Statex where \( E_j \) denotes the event \( E_j := \{ x_i \in B_{\epsilon_{i-1}}(\bar{x}_i) \text{ for all } i \in [0, j] \} \).
\EndFor
\Statex Generate a point \( x_{T+1} \) satisfying
\Statex \[ \mathbb{P} \left[ f^T(x_{T+1}) - \min f^T \leq \delta \mid E_T \right] \geq 1 - p. \]
\Return \( x_{T+1} \)
\end{algorithmic}
\end{algorithm}

Thus the \texttt{proxBoost} procedure consists of three stages, which we now examine in detail.

\textbf{Stage I: Initialization.} Algorithm 2 begins by generating a point \( x_0 \) that is a distance of \( \sqrt{\frac{2\delta}{\mu}} \) away from the minimizer of \( f \) with probability \( 1 - p \). This task can be achieved by applying a robust distance estimator on \( f \), as discussed previously.

\textbf{Stage II: Proximal iterations.} In each subsequent iteration, \( x_{j+1} \) is defined to be a point that is within a radius of \( \epsilon_j = \sqrt{\frac{2\delta}{\mu+\lambda_j}} \) from the minimizer of \( f^j \) with probability \( 1 - p \) conditioned on the event \( E_j \). The event \( E_j \) encodes that each previous iteration was successful in the sense that the point \( x_i \) indeed lies inside \( B_{\epsilon_{i-1}}(\bar{x}_i) \) for all \( i = 0, \ldots, j \). Thus \( x_{j+1} \) can be determined by a procedure that within the event \( E_j \) is a robust distance estimator on the function \( f^j \).
Stage III: Cleanup. In the final step, the algorithm outputs a $\delta$-minimizer of $f^T$ with probability $1 - p$ conditioned on the event $E_T$. In particular, if $f$ is $L$-smooth then we may use a robust distance estimator on $f^T$. Namely, taking into account the upper bound (2.1), we may declare $x_{T+1}$ to be any point satisfying

$$
P \left[ \|x_{T+1} - \bar{x}_{T+1}\| \leq \sqrt{\frac{2\delta}{L+\lambda_T}} \mid E_T \right] \geq 1 - p.
$$

Notice that by choosing $T$ sufficiently large, we may ensure that the condition number $\frac{\mu + \lambda_T}{L + \lambda_T}$ of $f^T$ is arbitrarily close to one. If $f$ is not smooth, such as when constraints are present, we cannot use a robust distance estimator in the cleanup stage. We will see in Section 5 a different approach, based on the robust stochastic gradient method of [22].

The following theorem summarizes the guarantees of the proxBoost procedure.

**Theorem 3.2 (Proximal Boost).** Fix a constant $\delta > 0$, a probability of failure $p \in (0, 1)$ and a natural number $T \in \mathbb{N}$. Then with probability at least $1 - (T + 2)p$, the point $x_{T+1} = \text{proxBoost}(\delta, p, T)$ satisfies

$$
f(x_{T+1}) - \min f \leq \delta \left( 1 + \sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}} \right).
$$

(3.6)

**Proof.** We first prove by induction the estimate

$$
P[E_t] \geq 1 - (t + 1)p \quad \text{for all } t = 0, \ldots, T.
$$

(3.7)

The base case $t = 0$ is immediate from the definition of $x_0$. Suppose now that (3.7) holds for some index $t - 1$. Then the inductive assumption and the definition of $x_t$ yield

$$
P[E_t] = P[E_t \mid E_{t-1}]P[E_{t-1}] \geq (1 - p)(1 - tp) \geq 1 - (t + 1)p,
$$

thereby completing the induction. Thus the inequalities (3.7) hold. Define the event

$$
F = \{ f^T(x_{T+1}) - \min f^T \leq \delta \}.
$$

We therefore deduce

$$
P[F \cap E_T] = P[F \mid E_T] \cdot P[E_T] \geq (1 - (T + 1)p)(1 - p) \geq 1 - (T + 2)p.
$$

Suppose now that the event $F \cap E_T$ occurs. Then using the estimate (3.2), we conclude

$$
f(x_{T+1}) - \min f \leq (f^T(x_{T+1}) - f^T(\bar{x}_{T+1})) + \sum_{i=0}^{T} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2 \leq \delta + \sum_{i=0}^{T} \frac{\delta \lambda_i}{\mu + \lambda_{i-1}},
$$

where the last inequality uses the definitions of $x_{T+1}$ and $\epsilon_j$. This completes the proof. \qed
Looking at the estimate \([3.6]\), we see that the final error \(f(x_{T+1}) - \min f\) is controlled by the sum \(\sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}}\). A moment of thought yields an appealing choice \(\lambda_i = \mu 2^i\) for the proximal parameters. Indeed, then every element in the sum \(\frac{\lambda_i}{\mu + \lambda_{i-1}}\) is upper bounded by two. Moreover, if \(f\) is \(L\)-smooth, then the condition number \(\frac{L + \lambda_T}{\mu + \lambda_T}\) of \(f^T\) is upper bounded by two after only \(T = \lceil \log(L/\mu) \rceil\) rounds.

**Corollary 3.3 (Proximal boost with geometric decay).** Fix an iteration count \(T\), a target accuracy \(\epsilon > 0\), and a probability of failure \(p \in (0, 1)\). Define the algorithm parameters:

\[
\delta = \frac{\epsilon}{2 + 2T} \quad \text{and} \quad \lambda_i = \mu 2^i \quad \forall i \in [0, T].
\]

Then the point \(x_{T+1} = \text{proxBoost}(\delta, p, T)\) satisfies

\[
P(f(x_{T+1}) - \min f \leq \epsilon) \geq 1 - (T + 2)p.
\]

In the next two sections, we seed the \text{proxBoost} procedure with (accelerated) stochastic gradient algorithms and methods based on empirical risk minimization. The reader, however, should keep in mind that \text{proxBoost} is entirely agnostic to the inner workings of the robust distance estimators it uses. The only point to be careful about is that some distance estimators (e.g. stochastic gradient) require to be passed auxiliary quantities, such as an upper estimate on the function gap at the initial point. Therefore, we may have to update such estimates along the iterations of Algorithm 2.

### 4 Consequences for empirical risk minimization

In this section, we explore the consequences of the \text{proxBoost} algorithm for empirical risk minimization. Setting the stage, fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and equip \(\mathbb{R}^d\) with the Borel \(\sigma\)-algebra. Consider the optimization problem

\[
\min_x f(x) = \mathbb{E}_{z \sim \mathbb{P}} [f(x, z)],
\]

where \(f : \mathbb{R}^d \times \Omega \to \mathbb{R}_+\) is a measurable nonnegative function. A common approach to expectation minimization problems is based on empirical risk minimization. Namely, we may form an i.i.d. sample \(z_1, \ldots, z_n \sim \mathbb{P}\) and minimize the empirical average

\[
\min_x f_S(x) := \frac{1}{n} \sum_{i=1}^{n} f(x, z_i).
\]

A central question is to determine the number \(n\) of samples that would ensure that the minimizer \(x_S\) of the empirical risk has low generalization error \(f(x_S) - \min f\), with reasonably high probability. There is a vast literature on this subject; some representative works include \([5, 19, 33, 39]\). We build here on the work of Hsu-Sabato \([19]\), who specifically focused on high confidence guarantees for smooth strongly convex minimization. As in the previous sections, we let \(\bar{x}\) be a minimizer of \(f\) and define the shorthand \(f^* = \min f\).
Assumption 4.1. Following [19], we make the following assumptions on the loss.

1. **(Strong convexity)** There exist a real $\mu > 0$ and a natural number $N \in \mathbb{N}$ such that:
   a) the population loss $f$ is $\mu$-strongly convex,
   b) the empirical loss $x \mapsto f_S(x)$ is $\mu$-strongly convex with probability at least $5/6$, whenever $|S| \geq N$.

2. **(Smoothness)** There exist constants $L, \hat{L} > 0$ such that:
   a) for a.e. $z \sim P$, the loss $x \mapsto f(x,z)$ is $\hat{L}$-smooth,
   b) the population objective $x \mapsto f(x)$ is $L$-smooth.

Define the empirical and population condition numbers, $\hat{\tau} := \hat{L}/\mu$ and $\tau = L/\mu$, respectively.

The following result proved in [19, Theorem 15] shows that the empirical risk minimizer is a weak distance oracle for the problem (4.1).

**Lemma 4.2.** Fix an i.i.d. sample $z_1, \ldots, z_n \sim P$ of size $n \geq N$. Then the minimizer $x_S$ of the empirical risk (4.2) satisfies the bound:

$$
\mathbb{P} \left[ \|x_S - \bar{x}\| \leq \sqrt{\frac{96\hat{L}f(x_*)}{n\mu^2}} \right] \geq 2/3.
$$

In particular, using Algorithm 1 we may turn empirical risk minimization into a robust distance estimator for the problem (4) using a total of $mn$ samples. Let us estimate the function value at the generated point by a direct application of smoothness. Appealing to Lemma 2.2 and (2.1), we deduce that with probability $1 - \exp(-m/18)$ the procedure will return a point $x$ satisfying

$$
f(x) \leq \left(1 + \frac{432\hat{L}L}{n\mu^2}\right)f^*.
$$

Observe that this is an estimate of relative error. In particular, let $p \in (0,1)$ be some acceptable probability of failure and let $\gamma > 0$ be a desired level of relative accuracy. Then setting $m = \lceil 18 \ln(1/p) \rceil$ and $n \geq \max\{\frac{432\hat{L}L}{\gamma\mu^2}, N\}$, we conclude that $x$ satisfies

$$
\mathbb{P}[f(x) \leq (1 + \gamma)f^*] \geq 1 - p,
$$

(4.3)

while the overall sample complexity of the procedure is

$$
\left\lceil 18 \ln \left(\frac{1}{p}\right) \right\rceil \cdot \max \left\{ \left\lceil \frac{432\hat{L}L}{\gamma}\right\rceil, N \right\}.
$$

(4.4)

This is exactly the result [19, Corollary 16]. We will now see how to find a point $x$ satisfying (4.3) with significantly fewer samples by embedding empirical risk minimization within
the \texttt{proxBoost} algorithm. Algorithm \texttt{3} encodes the empirical risk minimization process on a quadratically regularized problem. Algorithm \texttt{4} is the robust distance estimator induced by Algorithm \texttt{3}. Finally, Algorithm \texttt{5} is the \texttt{proxBoost} algorithm specialized to empirical risk minimization.

\begin{algorithm}[H]
\caption{ERM ($n, \lambda, x$)}
\begin{algorithmic}
\State \textbf{Input:} sample count $n \in \mathbb{N}$, center $x \in \mathbb{R}^d$, amplitude $\lambda > 0$.
\State Generate i.i.d. samples $z_1, \ldots, z_n \sim \mathcal{P}$ and compute the minimizer $\bar{y}$ of
\begin{equation*}
\min_y \frac{1}{n} \sum_{i=1}^{n} f(y, z_i) + \frac{\lambda}{2} \|y - x\|^2.
\end{equation*}
\State \textbf{Return} $\bar{y}$
\end{algorithmic}
\end{algorithm}

\begin{algorithm}[H]
\caption{ERM-R ($n, m, \lambda, x$)}
\begin{algorithmic}
\State \textbf{Input:} sample count $n \in \mathbb{N}$, trial count $m \in \mathbb{N}$, center $x \in \mathbb{R}^d$, amplitude $\lambda > 0$.
\State Query $m$ times ERM($n, \lambda, x$) and let $Y = \{y_1, \ldots, y_m\}$ consist of the responses.
\State \textbf{Step} $j = 1, \ldots, m$:
\State \hspace{1em} Compute $r_i = \min\{r \geq 0 : |B_r(y_i) \cap Y| > \frac{m}{2}\}$.
\State \hspace{1em} Set $i^* = \arg\min_{i \in [1, m]} r_i$
\State \textbf{Return} $y_{i^*}$
\end{algorithmic}
\end{algorithm}

\begin{algorithm}[H]
\caption{BoostERM ($\gamma, T, m$)}
\begin{algorithmic}
\State \textbf{Input:} $T, m \in \mathbb{N}$, $\gamma > 0$.
\State \hspace{1em} Set $\lambda_{-1} = 0$, $n_{-1} = \frac{432 L}{\gamma \mu}$.
\State \textbf{Step} $j = 0, \ldots, T$:
\State \hspace{1em} $x_j = \text{ERM-R}(n_{j-1}, m, \lambda_{j-1}, x_{j-1})$
\State \hspace{1em} $n_j = 432 \left[ \frac{L + \lambda_j}{\mu + \lambda_j} \left( \frac{1}{\gamma} + \sum_{i=0}^{j} \frac{\lambda_i}{\mu + \lambda_{i-1}} \right) \right] \vee N$
\State \textbf{Return} $x_{T+1} = \text{ERM-R}(\frac{L + \lambda_T}{\mu + \lambda_T} \cdot n_T, m, \lambda_T, x_T)$
\end{algorithmic}
\end{algorithm}

Using Theorem 3.2, we can now prove the following result.

\textbf{Theorem 4.3} (Efficiency of BoostERM). Fix a target relative accuracy $\gamma > 0$ and numbers $T, m \in \mathbb{N}$. Then with probability at least $1 - (T + 2) \exp\left(-\frac{m}{48}\right)$, the point $x_{T+1} = \text{BoostERM}(\gamma, T, m)$ satisfies

\begin{equation*}
f(x_{T+1}) - f^* \leq \left( 1 + \sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}} \right) \gamma f^*.
\end{equation*}

\textit{Proof.} We will verify that Algorithm \texttt{5} is an instantiation of Algorithm \texttt{2} with $\delta = \gamma f^*$ and $p = \exp\left(-\frac{m}{48}\right)$. More precisely, we will prove by induction that with this choice of $p$
and $\delta$, the iterates $x_j$ satisfy (3.4) for each index $j = 0, \ldots, T$ and $x_{T+1}$ satisfies (3.5). As the base case, consider the evaluation $x_0 = \text{ERM-R}(n_{-1}, m, \lambda_{-1}, x_{-1})$. Then Lemma 2.2 and Theorem 4.2 guarantee

$$
\mathbb{P} \left[ \|x_0 - \bar{x}_0\| \leq 3 \sqrt{\frac{96Lf^*}{n_{-1}\mu^2}} \right] \geq 1 - \exp \left( -\frac{m}{18} \right).
$$

Taking into account the definition of $n_{-1}$, we deduce

$$
\mathbb{P} \left[ \|x_0 - \bar{x}_0\| \leq \epsilon_{-1} \right] \geq 1 - p,
$$
as claimed. As an inductive hypothesis, suppose that (3.4) holds for the iterates $x_0, x_1, \ldots, x_{j-1}$. We will prove it holds for $x_j$. To this end, suppose that the event $E_{j-1}$ occurs. Then by the same reasoning as in the base case, the point $x_j$ satisfies

$$
\mathbb{P} \left[ \|x_j - \bar{x}_j\| \leq 3 \sqrt{\frac{96(\hat{L} + \lambda_{j-1})f^{j-1}(\bar{x}_j)}{n_{j-1}(\mu + \lambda_{j-1})^2}} \right] \geq 1 - \exp \left( -\frac{m}{18} \right). \quad (4.5)
$$

Observe now, using (3.1) and the inductive assumption, the estimate:

$$
f^{j-1}(\bar{x}_j) - f^* \leq \sum_{i=0}^{j-1} \frac{\lambda_i}{2} \|\bar{x}_i - x_i\|^2 \leq \delta \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}}.
$$

Combining this inequality with (4.5), we conclude that conditioned on the event $E_{j-1}$, we have with probability $1 - p$ the guarantee

$$
\frac{\mu + \lambda_{j-1}}{2} \|x_j - \bar{x}_j\|^2 \leq \frac{432(\hat{L} + \lambda_{j-1})(1 + \gamma \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}})}{n_{j-1}(\mu + \lambda_{j-1})} \cdot f^* = \gamma f^*, \quad (4.6)
$$

where the last equality follows from the definition of $n_{j-1}$. Thus the estimate (3.4) holds for the iterate $x_0, \ldots, x_T$, as needed. Suppose now that that event $E_T$ occurs. Then by exactly the same reasoning that led to (4.6), we have the estimate

$$
\frac{\mu + \lambda_T}{2} \|x_{T+1} - \bar{x}_{T+1}\|^2 \leq \frac{\mu + \lambda_T}{L + \lambda_T} \gamma f^*.
$$

Using smoothness, we therefore deduce $f^T(x_{T+1}) - \min f^T \leq \gamma f^* = \delta$, as claimed. An application of Theorem 3.2 completes the proof. \qed

When using the proximal parameters $\lambda_i = \mu 2^i$, we obtain the following guarantee.

**Corollary 4.4 (Efficiency of BoostERM with geometric decay).** Fix a target relative accuracy $\gamma' > 0$ and a probability of failure $p \in (0, 1)$. Define the algorithm parameters:

$$
T = \lceil \log_2(\tau) \rceil, \quad m = \left\lceil 18 \ln \left( \frac{T + 2}{p} \right) \right\rceil, \quad \gamma = \frac{\gamma'}{2 + 2T}, \quad \lambda_i = \mu 2^i.
$$

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Then with probability of at least $1 - p$, the point $x_{T+1} = \text{BoostERM}(\gamma, T, m)$ satisfies $f(x_{T+1}) \leq (1 + \gamma')f^*$. Moreover, the total number of samples used by the algorithm is

$$O \left( \ln(\tau) \ln \left( \frac{\ln(\tau)}{p} \right) \cdot \max \left\{ \left( 1 + \frac{1}{\gamma'} \right)^{\frac{1}{\tau}} \ln(\tau), N \right\} \right).$$

Notice that the sample complexity provided by Corollary 4.4 is an order of magnitude better than $4.4$ in terms of the dependence on the condition numbers $\hat{\tau}$ and $\tau$.

5 Consequences for stochastic approximation

In this section, we investigate the consequences of the proxBoost algorithm for stochastic approximation. Namely, we will seed proxBoost with the robust distance estimator, induced by the stochastic proximal gradient method and its accelerated variant. An important point is that the sample complexity of stochastic gradient methods depends on the initialization quality $f(x_0) - f^*$. Consequently, in order to know how many iterations are needed to reach a desired accuracy $\mathbb{E}f(x_i) - f^* \leq \delta$, we must have available an upper bound on the initialization quality $\Delta \geq f(x_0) - f^*$. Moreover, we will have to update the initialization estimate for each proximal subproblem along the iterations of proxBoost.

The following assumption formalizes this idea.

**Assumption 5.1.** Consider the proximal minimization problem

$$\min_y g(x) := f(y) + \frac{\lambda}{2}\|y - x\|^2,$$

Let $\Delta > 0$ be a real number satisfying $g(x) - \min g \leq \Delta$. We will let $\text{Alg}(\delta, \lambda, \Delta, x)$ be a procedure that returns a point $y$ satisfying

$$\mathbb{P}[g(y) - \min g \leq \delta] \geq \frac{2}{3}.$$ 

Clearly, by strong convexity, we may turn $\text{Alg}(\cdot)$ into a robust distance estimator on the proximal problems as long as $\Delta$ is indeed an upper bound on the initialization error. We record the robust distance estimator induced by $\text{Alg}(\cdot)$ as Algorithm 6.

**Algorithm 6: Alg-R($\delta, \lambda, \Delta, x, m$)**

<table>
<thead>
<tr>
<th><strong>Input:</strong> accuracy $\delta &gt; 0$, amplitude $\lambda &gt; 0$, upper bound $\Delta &gt; 0$, center $x \in \mathbb{R}^d$, trial count $m \in \mathbb{N}$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Query $m$ times $\text{Alg}(\delta, \lambda, \Delta, x)$ and let $Y = {y_1, \ldots, y_m}$ consist of the responses.</td>
</tr>
<tr>
<td><strong>Step</strong> $j = 1, \ldots, m$:</td>
</tr>
<tr>
<td>Compute $r_i = \min{r \geq 0 :</td>
</tr>
<tr>
<td>Set $i^* = \text{argmin}_{i \in [1, m]} r_i$</td>
</tr>
<tr>
<td><strong>Return</strong> $y_{i^*}$</td>
</tr>
</tbody>
</table>
When $f$ is $L$-smooth, it is straightforward to instantiate $\text{proxBoost}$ with the robust distance estimator $\text{Alg-R}$. The situation is more nuanced when $f$ is nonsmooth for two reasons. First, it becomes less clear how to control the initialization quality for each proximal subproblem. Secondly, $\text{Alg-R}$ can not be used in the cleanup stage of $\text{proxBoost}$ to generate the last iterate $x_{T+1}$; instead, we will use a different algorithm [22] in this last stage. In the following two sections, we consider the smooth and nonsmooth settings in order.

5.1 Smooth Setting

Throughout this section, in addition to Assumptions 2.1 and 5.1, we assume that $f$ is $L$-smooth and set $\tau = \frac{L}{\mu}$. Algorithm 7 seeds the $\text{proxBoost}$ procedure with $\text{Alg-R}$.

Algorithm 7: BoostAlg($\delta, \Delta, x, T, m$)

| Input: | accuracy $\delta > 0$, upper bound $\Delta > 0$, center $x \in \mathbb{R}^d$, iterations $m, T \in \mathbb{N}$ |
| Set $\lambda_{-1} = 0$, $\Delta_{-1} = \Delta$, $x_{-1} = x$ |
| Step $j = 0, \ldots, T$: |
| $x_j = \text{Alg-R}(\delta/9, \lambda_{j-1}, \Delta_{j-1}, x_{j-1}, m)$ |
| $\Delta_j = \delta \left( \frac{L + \lambda_{j-1}}{\mu + \lambda_{j-1}} + \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}} \right)$ |
| Return $x_{T+1} = \text{Alg-R}(\mu + \lambda_T, \frac{\delta}{9}, \lambda_T, \Delta_T, x_T, m)$ |

We can now prove the following theorem on the efficiency of Algorithm 7. The proof is almost a direct application of Theorem 3.2. The only technical point is to verify that for all indices $j$, the quantity $\Delta_j$ is a valid upper bound on the initialization error $f^j(x_j) - \min f^j$ in the event $E_j$.

**Theorem 5.2 (Efficiency of BoostAlg).** Fix an arbitrary point $x_{in} \in \mathbb{R}^d$ and let $\Delta_{in}$ be any upper bound $\Delta_{in} \geq f(x_{in}) - \min f$. Fix natural numbers $T, m \in \mathbb{N}$. Then with probability at least $1 - (T + 2) \exp \left( -\frac{m}{18} \right)$, the point $x_{T+1} = \text{BoostAlg}(\delta, \Delta_{in}, x_{in}, T, m)$ satisfies

$$f(x_{T+1}) - \min f \leq \delta \left( 1 + \sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}} \right).$$

**Proof.** We will verify that Algorithm 7 is an instantiation of Algorithm 2 with $p = \exp(-\frac{m}{18})$. More precisely, we will prove by induction that with this choice of $p$, the iterates $x_j$ satisfy (3.4) for each index $j = 0, \ldots, T$ and $x_{T+1}$ satisfies (3.5). To see the base case, observe that Lemma 2.2 guarantees that with probability $1 - p$, the estimate holds:

$$\|x_0 - \bar{x}_0\| \leq 3 \sqrt{\frac{2 \cdot \delta/9}{\mu}} = \epsilon_{-1}.$$ 

As an inductive hypothesis, suppose that (3.4) holds for the iterates $x_0, x_1, \ldots, x_{j-1}$. We will prove it holds for $x_j$. To this end, suppose that the event $E_{j-1}$ occurs. Then using
From (3.3) we deduce
\[
f^{j-1}(x_{j-1}) - \min f^{j-1} \leq f(x_{j-1}) - f^* \leq \frac{L + \lambda_{j-2}}{2} \|x_{j-1} - x_{j-1}\|^2 + \sum_{i=0}^{j-2} \frac{\lambda_i}{2} \|x_i - x_i\|^2
\]
\[
\leq \frac{\delta(L + \lambda_{j-2})}{\mu + \lambda_{j-2}} + \sum_{i=0}^{j-2} \frac{\delta\lambda_i}{\mu + \lambda_{i-1}} = \Delta_{j-1}, \quad (5.1)
\]
where the two inequalities follow from the inductive assumption and smoothness. Therefore Lemma 2.2 guarantees that conditioned on $E_{j-1}$ with probability $1 - p$, the estimate holds:
\[
\|x_j - \bar{x}_j\| \leq 3 \sqrt{\frac{2 \cdot \delta/9}{\mu + \lambda_{j-1}}} = \epsilon_{j-1}.
\]
Thus the estimate (3.4) holds for the iterate $x_j$, as needed.

Now suppose that the event $E_T$ holds. Then exactly the same reasoning that led to (5.3) yields the guarantee $f^T(x_T) - \min f^T \leq \Delta_T$. Therefore Lemma 2.2 guarantees that with probability $1 - p$ conditioned on $E_T$, we have
\[
\|x_{T+1} - \bar{x}_{T+1}\| \leq 3 \sqrt{\frac{2}{\mu + \lambda_T} \cdot \frac{\delta \cdot \mu + \lambda_T}{L + \lambda_T}} = \sqrt{\frac{2\delta}{L + \lambda_T}}.
\]
Taking into account smoothness of $L$, we therefore deduce
\[
\mathbb{P}[f^T(x_{T+1}) - \min f^T \leq \delta \mid E_T] \geq 1 - p,
\]
thereby establishing (3.5). An application of Theorem 3.2 completes the proof.

When using the proximal parameters $\lambda_i = \mu 2^i$, we obtain the following guarantee.

**Corollary 5.3 (Efficiency of BoostAlg with geometric decay).** Fix an arbitrary point $x_{in} \in \mathbb{R}^d$ and let $\Delta_{in}$ be any upper bound $\Delta_{in} \geq f(x_{in}) - \min f$. Fix a target accuracy $\delta' > 0$ and probability of failure $p \in (0, 1)$, and set the algorithm parameters
\[
T = \lceil \log_2(\tau) \rceil, \quad m = \left\lceil 18 \ln \left( \frac{2 + T}{p} \right) \right\rceil, \quad \delta = \frac{\delta'}{2 + 2T}, \quad \lambda_i = \mu 2^i.
\]
Then the point $x_{T+1} = \text{BoostAlg}(\delta, \Delta_{in}, x_{in}, T, m)$ satisfies
\[
\mathbb{P}(f(x_{T+1}) - \min f \leq \delta') \geq 1 - p.
\]
Moreover, the total number of calls to $\text{Alg}(\cdot)$ is
\[
\left\lceil 18 \ln \left( \frac{\left\lceil \log_2(\tau) \right\rceil}{p} \right) \right\rceil \left[ 2 + \log_2(\tau) \right],
\]
while the initialization errors satisfy
\[
\max_{i=0, \ldots, T+1} \Delta_i \leq \frac{\tau + 1 + 2 \left\lceil \log_2(\tau) \right\rceil}{2 + 2 \left\lceil \log_2(\tau) \right\rceil} \delta'.
\]
5.2 General Setting

In this section, we assume that $f$ is $\mu$-strongly convex, but we do not assume it is smooth. In the previous section, we used smoothness to explicitly construct a sequence $\{\Delta_j\}_j$, which bounded the initial functional error of the proximal subproblems. In the general setting, explicit upper bounds may be unavailable. Thus, we instead show that the iterates with high probability will never leave a small ball centered at the minimizer $\bar{x}$ of $f$. The radius of this ball is

$$\bar{\epsilon} := \sqrt{\frac{2\delta}{\mu}} + \sqrt{\frac{2\delta}{\mu} \sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}}}.$$  

In the proof, we will show the initial functional errors of the proximal subproblems are bounded by any constant $M_{\bar{\epsilon}}$, which satisfies

$$M_{\bar{\epsilon}} \geq \sup_{x \in B(\bar{x}) \cap \text{dom } f} \{f(x) - f^*\}.$$  

(5.2)

For example, if the objective function $f$ is $\ell$-Lipschitz continuous on $B(\bar{x}) \cap \text{dom } f$, then we may simply set $M_{\bar{\epsilon}} = \ell \bar{\epsilon}$. We should note that once we specialize to stochastic gradient methods, the quantity $M_{\bar{\epsilon}}$ will appear only logarithmically in the sample complexity.

We use the bound (5.2) in the following algorithm, reminiscent of Algorithm (7). We purposefully do not yet specify the procedure for obtaining $x_{T+1}$ from $x_T$.

**Algorithm 8: GBoostAlg($\delta, \Delta, x, T, m$)**

**Input:** accuracy $\delta > 0$, upper bound $\Delta > 0$, center $x \in \mathbb{R}^d$, iterations $m, T \in \mathbb{N}$
Set $\lambda_{-1} = 0$, $\Delta_{-1} = \Delta$, $x_{-1} = x$
Set $x_0 = \text{Alg-R}(\delta / 9, \lambda_{-1}, \Delta_{-1}, x_{-1}, m)$

**Step** $j = 1, \ldots, T$:
- $x_j = \text{Alg-R}(\delta / 9, \lambda_{j-1}, M_{\bar{\epsilon}}, x_{j-1}, m)$

**Return** Any point $x_{T+1}$ satisfying (3.5).

The following theorem is almost a direct consequence of Theorem 3.2. Following the notation of Algorithm 2, we define the events $E_j := \{x_i \in B_{\epsilon_i}(\bar{x}) \text{ for all } i \in [0, j]\}$, where $x_1, \ldots, x_T$ are the iterates generated by GBoostAlg.

**Theorem 5.4 (Efficiency of GBoostAlg).** Fix an arbitrary point $x_{in} \in \mathbb{R}^d$ and let $\Delta_{in}$ be any upper bound $\Delta_{in} \geq f(x_{in}) - \min f$. Fix natural numbers $T, m \in \mathbb{N}$ and consider the iterates $x_0, \ldots, x_{T+1}$ generated by the algorithm GBoostAlg($\delta, \Delta_{in}, x_{in}, T, m$). Then in the event $E_T$, all the iterates $x_0, \ldots, x_T$ lie in $B(\bar{x})$. Moreover, with probability at least $1 - (T + 2) \exp\left(-\frac{m}{18}\right)$, the last iterate $x_{T+1}$ satisfies

$$f(x_{T+1}) - \min f \leq \delta \left(1 + \sum_{i=0}^{T} \frac{\lambda_i}{\mu + \lambda_{i-1}}\right).$$

**Proof.** As before, we verify that Algorithm 7 is an instantiation of Algorithm 2 with $p = \exp(-\frac{m}{18})$. More precisely, we prove by induction that for $j = 0, \ldots, T$, the iterate
$x_j$ satisfies (3.4) and the inclusion $x_j \in B_{\bar{\epsilon}}(\bar{x})$ holds within the event $E_j$. To see the base case, observe that Lemma 2.2 guarantees that with probability $1 - p$, the estimate holds:

$$\|x_0 - \bar{x}_0\| \leq 3\sqrt{\frac{2 \cdot \delta / 9}{\mu}} = \epsilon_{-1}.$$  

Moreover, in the event $E_0$, we have $\|x_0 - \bar{x}\| \leq \epsilon_{-1} \leq \bar{\epsilon}$. As an inductive hypothesis, suppose that (3.4) holds for the iterates $x_0, x_1, \ldots, x_{j-1}$ and that $x_0, x_1, \ldots, x_{j-1}$ lie in the ball $B_{\bar{\epsilon}}(\bar{x})$ in the event $E_{j-1}$. We first verify that $x_j$ satisfies (3.4). To this end, suppose that the event $E_{j-1}$ occurs. Then by the inductive assumption, the inclusion $x_{j-1} \in B_{\bar{\epsilon}}(\bar{x})$ holds. Therefore we deduce

$$f^{j-1}(x_{j-1}) - \min f^{j-1} \leq f(x_{j-1}) - f^* \leq M_{\epsilon}, \quad (5.3)$$

Thus Lemma 2.2 guarantees that conditioned on $E_{j-1}$ with probability $1 - p$, the estimate holds:

$$\|x_j - \bar{x}_j\| \leq 3\sqrt{\frac{2 \cdot \delta / 9}{\mu + \lambda_{j-1}}} = \epsilon_{j-1}.$$  

We conclude that the estimate (3.4) holds for the iterate $x_j$, as needed.

Next, suppose that the event $E_j$ holds. To prove the inclusion $x_j \in B_{\bar{\epsilon}}(\bar{x})$, observe that by strong convexity and the inequality (3.1), we have the estimate

$$\frac{\mu}{2}\|\bar{x}_j - \bar{x}\|^2 \leq f(\bar{x}_j) - f^* \leq f^{j-1}(\bar{x}_j) - f^* \leq \sum_{i=0}^{j-1} \frac{\lambda_i}{2}\|\bar{x}_j - x_i\|^2.$$  

Rearranging and using the inductive assumption, we obtain the bound

$$\|\bar{x}_j - \bar{x}\| \leq \sqrt{\frac{2 \sum_{i=0}^{j-1} \lambda_i}{\mu \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}}} \|\bar{x}_j - x_i\|^2} \leq \sqrt{\frac{2 \delta}{\mu \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}}} \|\bar{x}_j - x_i\|^2}.$$  

Consequently, the triangle inequality yields

$$\|x_j - \bar{x}\| \leq \|x_j - \bar{x}_j\| + \|\bar{x}_j - \bar{x}\| \leq \sqrt{\frac{2 \delta}{\lambda_{j-1} + \mu}} + \sqrt{\frac{2 \delta}{\mu \sum_{i=0}^{j-1} \frac{\lambda_i}{\mu + \lambda_{i-1}}} \leq \bar{\epsilon},}$$

as desired. This completes the induction. An application of Theorem 3.2 completes the proof.

When using the proximal parameters $\lambda_i = \mu 2^i$, we obtain the following guarantee.

**Corollary 5.5** (Efficiency of GBoostAlg with geometric decay). Fix an arbitrary point $x_{in} \in \mathbb{R}^d$ and let $\Delta_{in}$ be any upper bound $\Delta_{in} \geq f(x_{in}) - \min f$. Fix a target accuracy $\delta' > 0$ and probability of failure $p \in (0, 1)$, and set the algorithm parameters

$$T = \lceil \log_2(\tau) \rceil, \quad m = \left\lceil 18 \ln \left(\frac{2 + T}{p}\right) \right\rceil, \quad \delta = \frac{\delta'}{2 + 2T}, \quad \lambda_i = \mu 2^i.$$  

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Consider the iterates $x_0, \ldots, x_{T+1}$ generated by the algorithm $\text{GBestAlg}(\delta, \Delta_{\text{in}}, x_{\text{in}}, T, m)$. Then in the event $E_T$, all the iterates $x_1, x_2, \ldots, x_T$ lie in the ball $B_{\sqrt{2\delta'/\mu}(\bar{x})}$. Moreover, the last iterate $x_{T+1}$ satisfies

$$\mathbb{P}(f(x_{T+1}) - \min f \leq \delta') \geq 1 - p.$$ 

The total number of calls to $\text{Alg}(\cdot)$ is

$$\left\lceil 18 \ln \left( \frac{[2 + \log_2(\tau)]}{p} \right) \right\rceil [2 + \log_2(\tau)].$$

To illustrate Corollaries 5.3 and (5.5), we now specialize the result to the setting when $\text{Alg}(\cdot)$ is the stochastic proximal gradient method and its accelerated variant. This application is only meant to be an illustration; indeed, $\text{proxBoost}$ can be applied to other streaming algorithms, as well. For example, by exactly the same reasoning, one can couple $\text{proxBoost}$ with variance reduced methods for minimizing finite sums of expectations [25].

**Illustration: robust (accelerated) stochastic gradient methods**

Consider the optimization problem

$$\min_{x \in \mathcal{X}} \varphi(x) + \psi(x)$$

where $\mathcal{X}$ is a closed convex set, the loss $\varphi : \mathbb{R}^d \to \mathbb{R}$ is convex and $L$-smooth, the regularizer $\psi : \mathbb{R}^d \to \mathbb{R}$ is continuous and convex, and $\varphi + \psi$ is strongly convex on $\mathcal{X}$. We denote the condition number of this problem by $\tau = L/\mu$. We also define the function $f = \varphi + \psi(x) + \iota_{\mathcal{X}}$, where $\iota_{\mathcal{X}}$ is a function that is 0 on $\mathcal{X}$ and $+\infty$ off of it. Following the standard literature on streaming algorithms, we suppose that the only access to $\varphi$ is through a stochastic gradient oracle. Namely, fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $G : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a measurable map satisfying

$$\mathbb{E}_z G(x, z) = \nabla \varphi(x) \quad \text{and} \quad \mathbb{E}_z \|G(x, z) - \nabla \varphi(x)\|^2 \leq \sigma^2.$$ 

We suppose that for any point $x$, we may sample $z \sim \mathbb{P}$ and compute the vector $G(x, z)$, which serves as an unbiased estimator of the gradient $\nabla \varphi(x)$. The performance of standard numerical methods within this model of computation is judged by their sample complexity—the number of stochastic gradient evaluations $G(x, z)$ with $z \sim \mathbb{P}$ required by the algorithm to produce an approximate minimizer of the problem.

Fix an initial point $x_{\text{in}}$ and let $\Delta_{\text{in}} > 0$ satisfy $\Delta_{\text{in}} \geq f(x_0) - f^*$. It is well known that an appropriately modified proximal stochastic gradient method can generate a point $x$ satisfying $\mathbb{E}f(x) - f^* \leq \epsilon$ with sample complexity

$$O \left( \tau \log \left( \frac{\Delta_{\text{in}}}{\epsilon} \right) + \frac{\sigma^2}{\mu \epsilon} \right). \quad (5.4)$$
The accelerated stochastic gradient method of [17, Multi-stage AC-SA, Proposition 7] and the simplified optimal algorithm of [24, Restarted Algorithm C, Corollary 9] have the substantially better sample complexity

\[ O \left( \sqrt{\tau \log \left( \frac{\Delta \ln \epsilon}{\epsilon \ln \tau} \right)} + \frac{\sigma^2}{\mu \epsilon} \right). \]

Clearly, we may use either of these two procedures as \( \text{Alg}(\cdot) \) within the \text{proxBoost} framework.

**Smooth setting.** Suppose that \( \psi \equiv 0 \) and \( \mathcal{X} = \mathbb{R}^d \). Then using Corollary 5.3, we deduce that the two resulting algorithms will find a point \( x \) satisfying

\[ \mathbb{P}[f(x) - f^* \leq \epsilon] \geq 1 - p \]

with sample complexity

\[ O \left( \ln (\tau) \ln \left( \frac{\ln \tau}{p} \right) \cdot \ln \left( \frac{\Delta \ln \tau}{\epsilon} \right) \right), \]

and

\[ O \left( \ln (\tau) \ln \left( \frac{\ln \tau}{p} \right) \cdot \sqrt{\tau \ln \left( \frac{\Delta \ln \tau}{\epsilon} \right)} \right), \]

for the unaccelerated and accelerated methods, respectively. Thus, \text{proxBoost} endows the stochastic gradient method and its accelerated variant with high confidence guarantees at an overhead cost that is only polylogarithmic in \( \tau \) and logarithmic in \( 1/p \).

**General setting.** Let us return to the general case. Unlike the smooth setting, we have not yet specified how to perform the last step of \text{GBoostAlg}, that is how to obtain the point \( x_{T+1} \). For this purpose, we will apply the Robust Stochastic Mirror Descent (RMSD) algorithm of [22, Section 6] to the problem of minimizing \( f^T \). Roughly speaking, this algorithm will find an \( \epsilon \)-optimal point in function value with probability \( 1 - p \) using a similar number of samples, up to multiplication by \( \log(1/p) \), as the unaccelerated stochastic gradient method. Recall that \( f^T \) is smooth and strongly convex with parameters \( 2L \) and \( \mu + L \), respectively. Hence the sample complexity of this last step is dominated by \( \frac{\sigma^2}{\epsilon L} \log(1/p) \), which is negligible compared to the overall cost of the previous iterations of \text{GBoostAlg}.

We now explain the application of RMSD more formally. Set \( \delta, \delta', \lambda, m, \) and \( T \) according to Corollary 5.5. Unlike the previous algorithms of this paper, RMSD requires an estimate on the distance of the initial iterate \( x_T \) to the minimizer \( \bar{x}_{T+1} \). Using Corollary 5.5, we have such an estimate. Namely, within event \( E_T \), we compute:

\[ \frac{\mu + \lambda T}{2} \|x_T - \bar{x}_{T+1}\|^2 \leq f^T(x_T) - f^T(x^*) \leq f^T(x_T) - f^* \leq M \sqrt{2\delta'/\mu}. \]
Rearranging, we deduce
\[ \| x_T - \bar{x}_{T+1} \| \leq r_0 := \sqrt{\frac{M \sqrt{2\delta'/\mu}}{\mu + \lambda_T}}. \]

We will now apply [22, Theorem 3] with appropriate parameter settings. Fix a real number \( \gamma > 1/2 \) and define \( N \) to be the smallest integer satisfying
\[ N \geq \max\left\{ 4 \log_2 \left( \frac{2Lr_0}{C\delta} \right), \frac{2\gamma}{C'} \log \left( \frac{2Lr_0^2}{C\delta} / \frac{\sigma^2}{C'\delta} \right) \right\}, \]
where the constants \( C \) and \( C' \) are the same as the ones appearing in [22, Theorem 3]. Without loss of generality we may suppose that the last term in the maximum is dominant and that \( N = \frac{\sigma^2}{C'\delta} \). Then [22, Theorem 3] guarantees that the procedure will output a point \( y \) satisfying
\[ \mathbb{P}[f^T(y) - \min f^T \leq \delta] \geq 1 - C \log \left( \frac{4LC'r_0^3}{C\delta} \right) \exp(-\gamma), \]
with the number of samples that scales as \( \frac{\sigma^2}{L\delta'} \cdot \log \left( \frac{4L^2C'r_0^3N}{\sigma^2} \right) \). Therefore, it follows that we may obtain a point \( y \) satisfying
\[ \mathbb{P}[f^T(y) - \min f^T \leq \delta] \geq 1 - p, \]
using on the order of
\[ \frac{\sigma^2 \ln(\tau)}{L\delta'} \log \left( \frac{Lr_0^3 \ln(\tau)}{\delta'} \right) \log \left( \frac{\ln(\tau)}{p} \right) \]
samples. Therefore, appealing to Corollary [5.5] we deduce the following. If within \( \text{GBoostAlg}(\delta, \Delta_m, x_{in}, T, m) \), we use the proximal accelerated stochastic gradient method as \( \text{Alg}(\cdot) \) to generate \( x_0, x_1, \ldots, x_T \) and use RMSD to generate \( x_{T+1} \), then we can be sure that the returned point \( x_{T+1} \) satisfies \( f(x_{T+1}) - f^* \leq \delta' \) with probability \( 1 - p \). Moreover the number of samples used is on the order of
\[ \mathcal{O}\left( \frac{\sigma^2 \ln(\tau)}{L\delta'} \log \left( \frac{M \sqrt{2\delta'/\mu} \ln(\tau)}{\delta'} \right) \log \left( \frac{M \sqrt{2\delta'/\mu} \ln(\tau)}{\delta'} \right) \right) \].

Thus, \( \text{GBoostAlg} \) combined with RMSD in the last step endows the accelerated stochastic proximal gradient method with high confidence guarantees at an overhead cost that is only polylogarithmic in \( \tau, 1/\delta', M \sqrt{2\delta'/\mu}, \) and logarithmic in \( 1/p \).
References


