Geometry of orthogonally invariant matrix varieties

Dmitriy Drusvyatskiy
Mathematics, University of Washington

Joint work with
H.-L. Lee (UW), G. Ottaviano (Florence), and R.R. Thomas (UW)

UC Davis
Algebra & Discrete Mathematics

AFOSR YIP

"The Euclidean distance degree of orthogonally invariant matrix varieties", D-Lee-Ottaviani-Thomas,
To appear in **Israel J. Math.**, 2016.

"Counting real critical points of the distance to orthogonally invariant matrix sets", D-Lee-Thomas,
SIAM J. Matrix Anal. Applic. 36(3):1360-1380, 2015.

Absolute symmetry

Signed permutation group:

$$\Pi^n_\pm = \{\text{signed permutations}\}$$

Subset $S \subset \mathbf{R}^n$ is $\Pi^n_\pm$-invariant if

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Real orthogonal group:

$$O^n = \{n \times n \text{ orthogonal matrices}\}$$

Subset $\mathcal{M} \subset \mathbf{R}^{n \times n}$ is O^n-invariant if

$$U\mathcal{M} = \mathcal{M}U = \mathcal{M} \qquad \text{for all} \quad U \in O^n$$

Universal examples

Singular values decomp.: any $X \in \mathbf{R}^{n\times n}$ can be written as

$$X = U \begin{bmatrix} \sigma_1(X) & & & \\ & \sigma_2(X) & & \\ & & \ddots & \\ & & & \sigma_n(X) \end{bmatrix} V^T$$

with $U, V \in O^n$ and $\sigma_1(X) \geq \sigma_2(x) \geq \ldots \geq \sigma_n(X) \geq 0$.

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Examples of O^n-invariant sets:

$$\mathbf{R}_r^{n\times n} = \{X : \operatorname{rank} X \leq r\} \qquad \mathbf{B}_* = \{X : \sum_i \sigma_i(X) \leq 1\}$$

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Concisely $S = \{x \in \mathbf{R}^n : \operatorname{Diag}(x) \in \mathcal{M}\}$ is diagonal restriction

Guiding philosophy

Transfer Principle:

“Properties” of $\sigma^{-1}(S)$ and S are in one-to-one correspondence.

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  Reason: $e_i(\sigma_1^2(X), \ldots, \sigma_n^2(X))$ coefficients of $\det(\lambda I - X^T X)$.

Metric comparison: (von Neumann '37, Fan '49)
Always

$$\|\sigma(X) - \sigma(Y)\|_2 \leq \|X - Y\|_F.$$

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$$U^T X V = \operatorname{Diag} \sigma(X) \quad \text{and} \quad U^T Y V = \operatorname{Diag} \sigma(Y)$$

(simultaneous ordered singular value decomp.)

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Special case: (Hardy-Littlewood-Pólya '52)

$$\|x^{\uparrow} - y^{\uparrow}\| \leq \|x - y\|$$

Euclidean Distance Degree

Consider variety

$$\mathcal{V} = \{x \in \mathbf{R}^n : f_1(x) = f_2(x) = \ldots = f_k(x) = 0\}.$$

with f_i polynomials.

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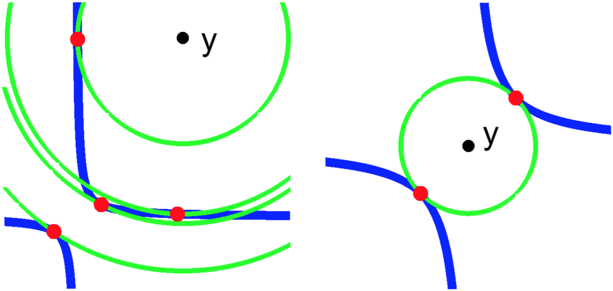
Defn (ED critical point):

$$x \in \text{EDcrit}(y) \quad \Longleftrightarrow \quad x \text{ smooth on } \mathcal{V}, \quad y - x \perp \mathcal{T}_x\mathcal{V}_{\mathbb{C}}.$$



$\boxed{\text{EDdegree}(\mathcal{V}) = |\text{EDcrit}(y)|}$ (constant for general y!).

(Draisma-Horobeţ-Ottaviani-Sturmfels-Thomas '15)

Hilbert & Cohn-Vossen:
"Anschauliche Geometrie" (Geometry and the Imagination)
Springer-Verlag, Berlin 1932

"The simplest curves are the planar curves.
Among them the simplest one is the line.
The next simplest is the circle.
After that comes the parabola,
and finally, general conics."

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Among them the simplest one is the line (EDdegree 1).
The next simplest is the circle (EDdegree 2).
After that comes the parabola (EDdegree 3),
and finally, general conics (EDdegree 4).

ED degree of O^n-invariant varieties

Theorem: (D-Lee-Ottaviani-Thomas '15)

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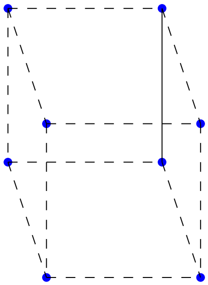
Examples:

Orthogonal Group: $O^n = \{X : X^T X = I\}$

$$\text{EDdegree}(O^n) = 2^n$$



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ED degree of O^n-invariant varieties

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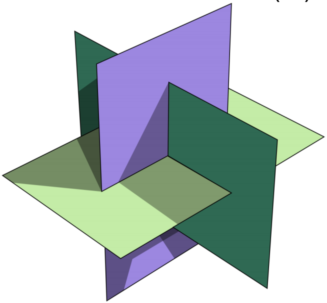
Examples:

Low rank matrices: $\mathbf{R}_r^{n\times n} = \{X : \operatorname{rank} X \le r\}$

$$\mathrm{EDdegree}(\mathbf{R}_r^{n\times n}) = \binom{n}{r}$$



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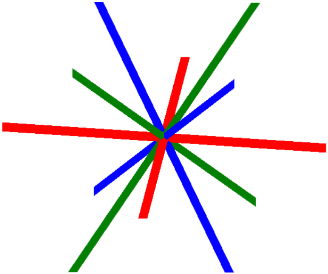
Examples:

Essential Variety: $\mathcal{E}^{3\times 3} = \{X : \sigma_1(X) = \sigma_2(X), \sigma_3(X) = 0\}$

$$\text{EDdegree}(\mathcal{E}^n) = 6$$



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ED degree of O^n-invariant varieties

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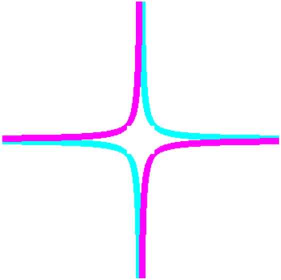
Examples:

Special Linear Group: $\quad \mathrm{SL}^n_{\pm} = \{X : \det(X) = \pm 1\}$

$$\text{EDdegree}(\mathrm{SL}^n_{\pm}) = n2^n$$



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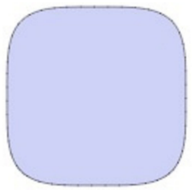
Examples:

Schatten hypersurface: $\mathcal{F}_{n,d} = \{X : \sum_i \sigma_i(X)^d = 1\}$

$$\mathrm{EDdegree}(\mathcal{F}_{n,d}) = d\sum_{i=1}^{n}(d-1)^i - \sum_{j=1}^{n-1}\binom{n}{j+1}\delta(j, d-2).$$



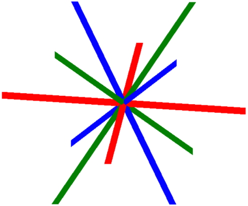
(Lee '14)

Why is this surprising?

Main challenges:

1) No explicit polynomial description of $\sigma^{-1}(S)$.

Eg. $\mathcal{E}^{3\times 3} = \{X : \det(X) = 0,\ 2XX^TX - \mathrm{tr}\,(XX^T)X = 0\}$



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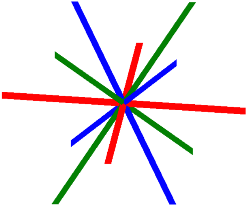
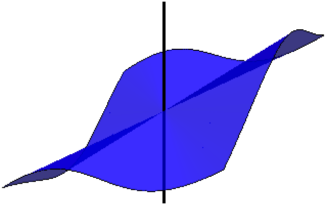


2) Smoothness is finicky:

Eg. Cartan Umbrella

$$\mathcal{V} = \{(x, y, z) : z(x^2 + y^2) - x^3 = 0\}$$



Strategy: step I (complexification)

Setting $\mathcal{M} = \sigma^{-1}(S)$, we have

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Obstruction: (Choudhury-Horn '87)
A matrix $X \in \mathbb{C}^{n \times n}$ admits an algebraic SVD

$$X = U\operatorname{Diag}(s)V^T \quad \text{with } U, V \in O^n_{\mathbb{C}} \text{ and } s \in \mathbb{C}^n$$

if and only if XX^T is diagonalizable and $\operatorname{rank} X = \operatorname{rank} XX^T$.

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Thm: (D-Lee-Ottaviani-Thomas '15)
Equalities hold:

$$\mathcal{S}_{\mathbb{C}} = \{x : \mathrm{Diag}\,(x) \in \mathcal{M}_{\mathbb{C}}\} \quad \text{and} \quad \mathcal{M}_{\mathbb{C}} = \overline{\bigcup_{U,V \in O^n_{\mathbb{C}}} U\,\mathrm{Diag}\,(S_{\mathbb{C}})\,V^T}$$

Strategy: step II (GIT perspective)

Suppose $\mathcal{M}_{\mathbb{C}}$ is invariant under $\mathcal{G} := O_{\mathbb{C}}^n \times O_{\mathbb{C}}^n$ as before.

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Then $\text{Inv} \cong \mathbb{C}[\mathcal{M}_{\mathbb{C}} /\!\!/ \mathcal{G}]$ where $\mathcal{M}_{\mathbb{C}} /\!\!/ \mathcal{G}$ is the *GIT quotient*

$$\mathcal{M}_{\mathbb{C}} /\!\!/ \mathcal{G} = \{y \in \mathbb{C}^n : y_i = e_i(x_1^2, \ldots, x_n^2) \quad \text{for some } x \in S_{\mathbb{C}}\}$$

(Geometric Invariant Theory (GIT) developed by Mumford '65)

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Then the quotient map is

$$\begin{aligned} \pi \colon \mathcal{M}_{\mathbb{C}} &\to \mathcal{M}_{\mathbb{C}} /\!\!/ \mathcal{G} \\ X &\mapsto (e_1(XX^T), \ldots, e_n(XX^T)) \end{aligned}$$

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- The fiber $\pi^{-1}(y)$ is a union of orbits, and
- $\pi^{-1}(y)$ is a single orbit $\Leftrightarrow$ $\pi^{-1}(y)$ contains a diagonal matrix.

Conclusion

- Classical ideas in algebra/geometry/matrix analysis come together to explain behavior of O^n-invariant varieties.

Thank you.