Slope and geometry in variational mathematics

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Joint work with
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Oct 5, 2013

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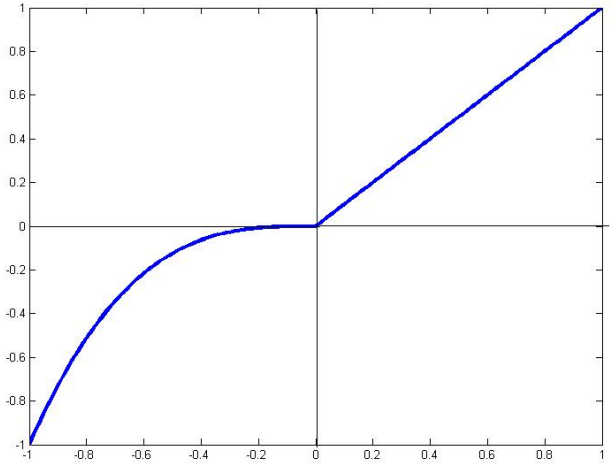

Method of alternating projections

Common problem:

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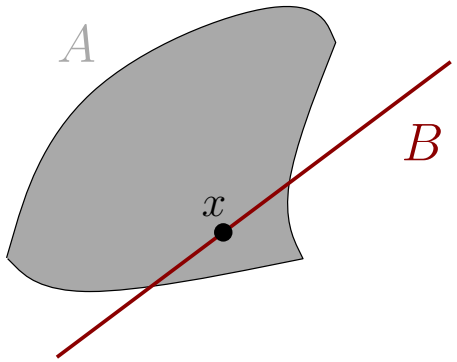

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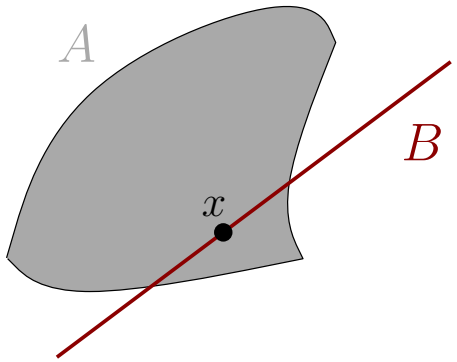

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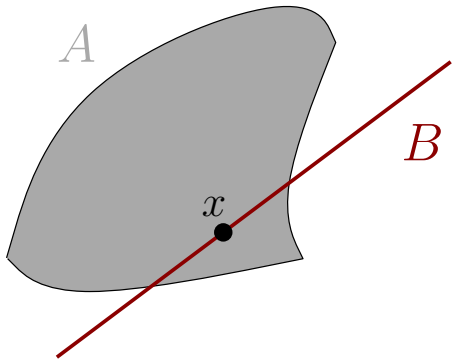

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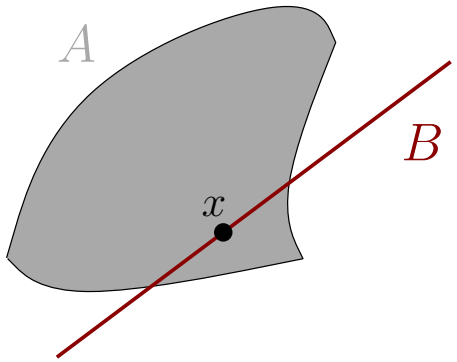

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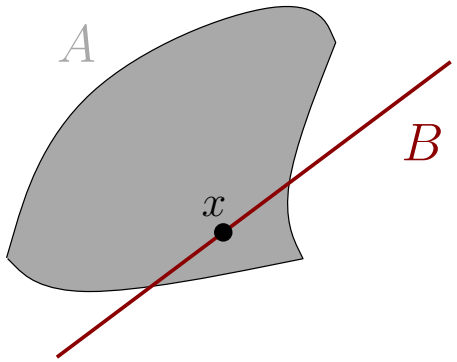

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Eg 2 (more interesting): Low-order control:

$$\{X \succeq 0 : \operatorname{rank} X \leq r\} \bigcap \{X : \mathcal{A}(X) = b\}.$$

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Method of alternating projections (von Neumann '33):

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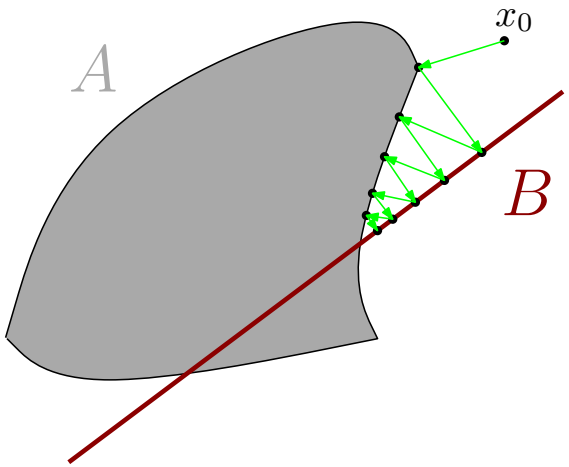

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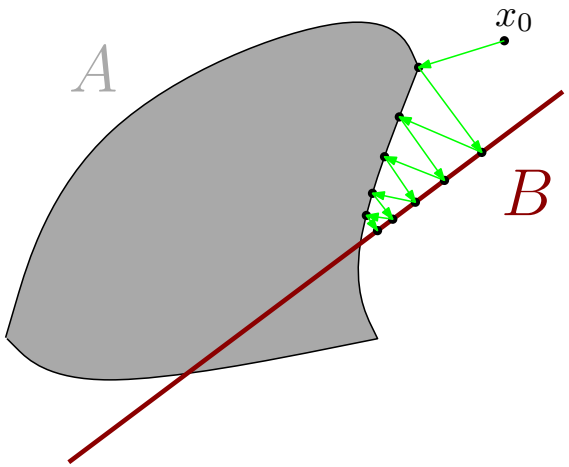

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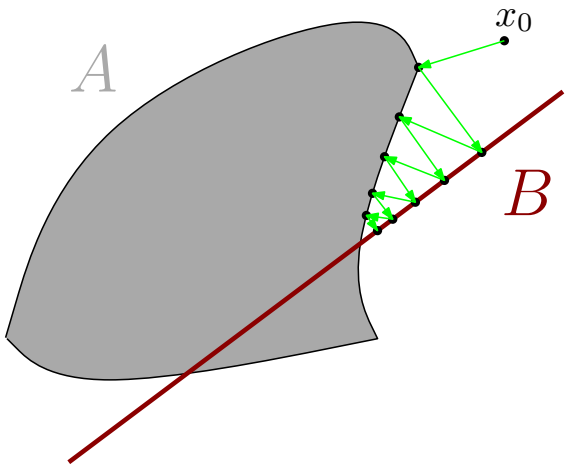

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Quantifying the angle:

$$\psi(x,y) := \begin{cases} |x-y| & \text{if } x \in A, y \in B \\ +\infty & \text{otherwise} \end{cases}$$

Comparison of $|\nabla\psi_y|(x)$ and $|\nabla\psi_x|(y)$ quantifies the angle!

Outline

- **When is slope an adequate tool?**
  - Semi-algebraic case
  - Slope & error bounds

- **Applications:**
  - Alternating projections & transversality
  - Steepest descent curves
  - Active sets & generic properties.

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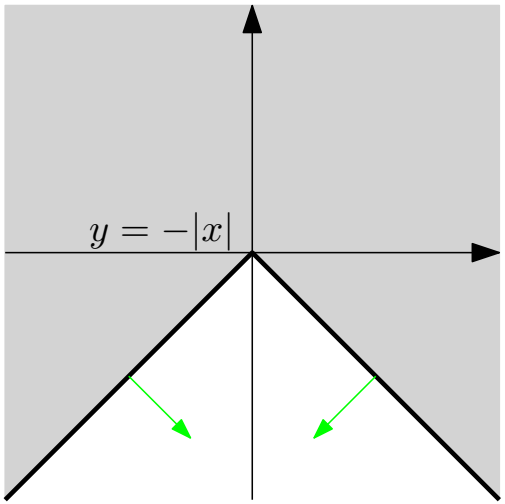

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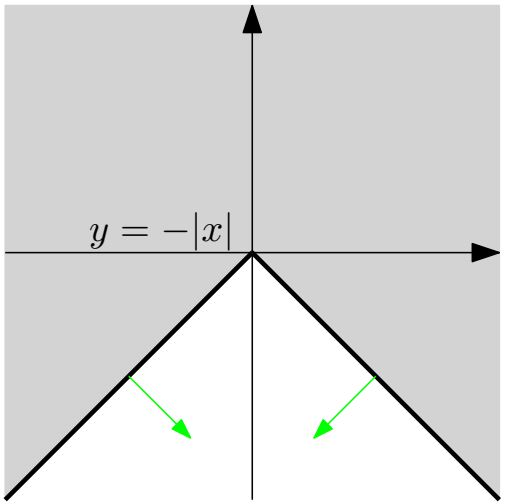

Subdifferential graph:

$$\boxed{\operatorname{gph} \partial f := \{(x, v) \in \mathbf{R}^n \times \mathbf{R}^n : v \in \partial f(x)\}}$$

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What do we expect the size of $\operatorname{gph} \partial f$ to be?

- If $f \colon \mathbf{R}^n \to \mathbf{R}$ is smooth, then $\operatorname{gph} \partial f$ is n-dimensional smooth manifold.
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Multiple authors (Rockafellar, Borwein, Wang,...):
There are functions $f\colon \mathbf{R}^n \to \mathbf{R}$ with "$2n$-dimensional" $\operatorname{gph} \partial f$.

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Semi-algebraic Q "stratify" into finitely many manifolds $\{\mathcal{M}_i\}$.
Dimension:

$$\dim Q := \max_{i=1,\ldots,k} \{\dim \mathcal{M}_i\}.$$

Semi-algebraic geometry

Theorem (D-Ioffe-Lewis)

For semi-algebraic $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, *we have*

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What can we learn from non-criticality?

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- Observed by Azé-Corvellec '04, Ioffe '00.

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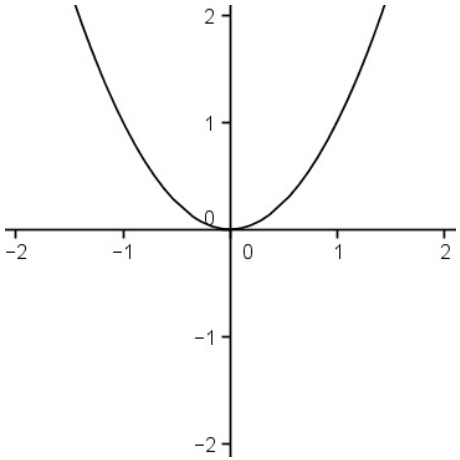

Figure: $f(x) = x^2$

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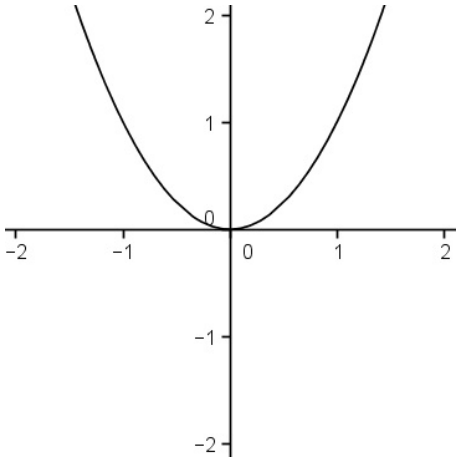

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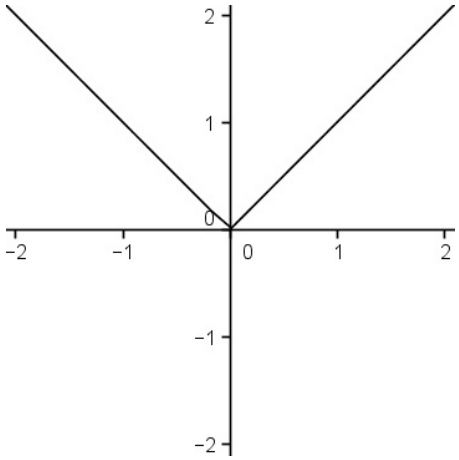

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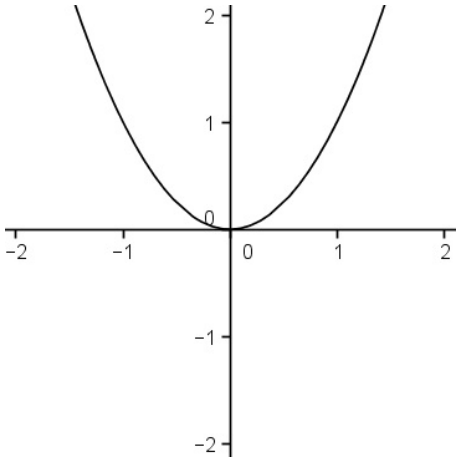

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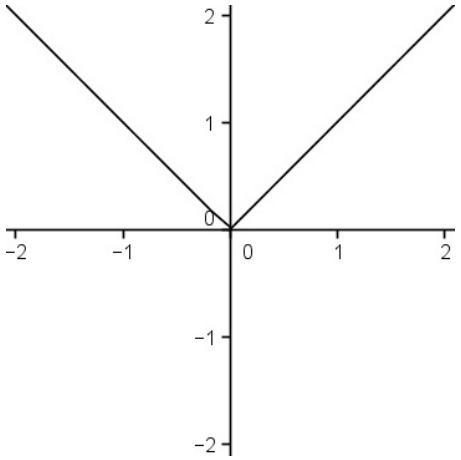

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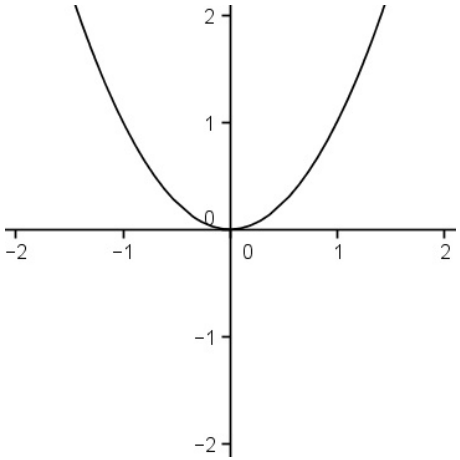

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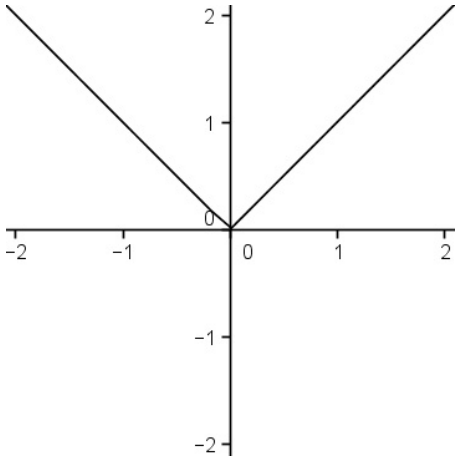

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Error bounds always applicable for semi-algebraic functions!

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Theorem (D-Ioffe)

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Quadratic growth:

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Second-order growth/Regularity: Poliquin-Rockafellar '98, D-Nghia-Mordukhovich '13, Artacho-Geoffroy '08 '13

Alternating projections & transversality

Convergence of alternating projections

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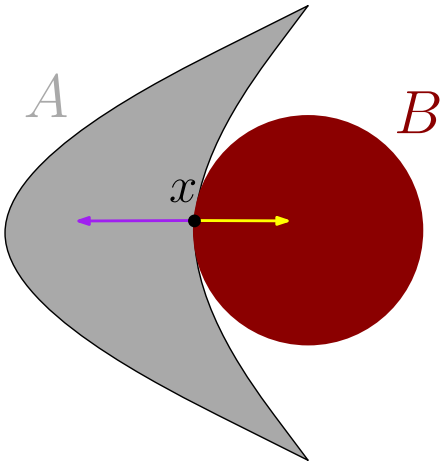
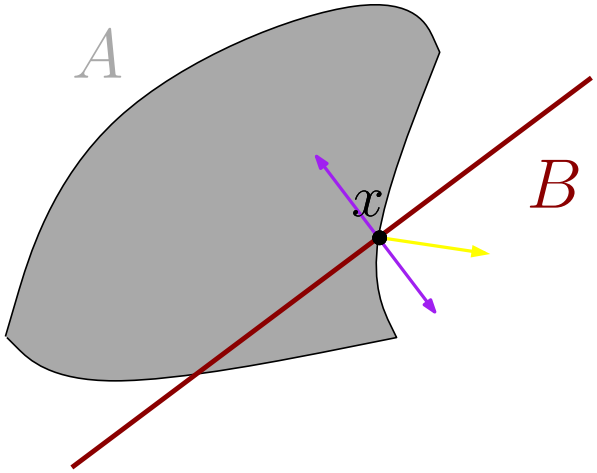


Figure: Not transverse



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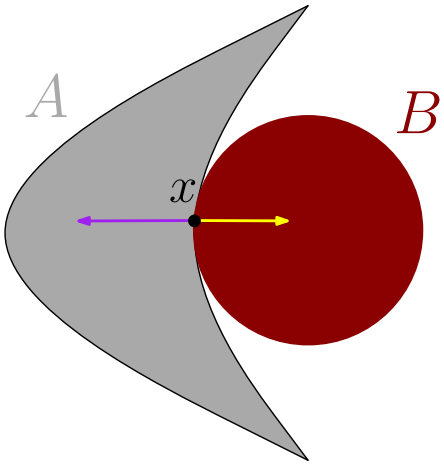


Figure: Not transverse

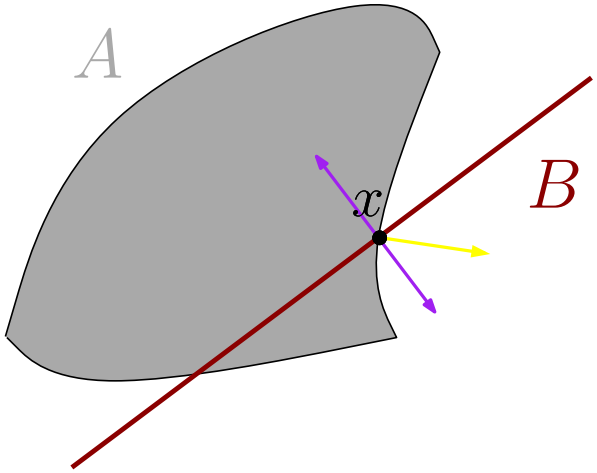


Figure: Transverse

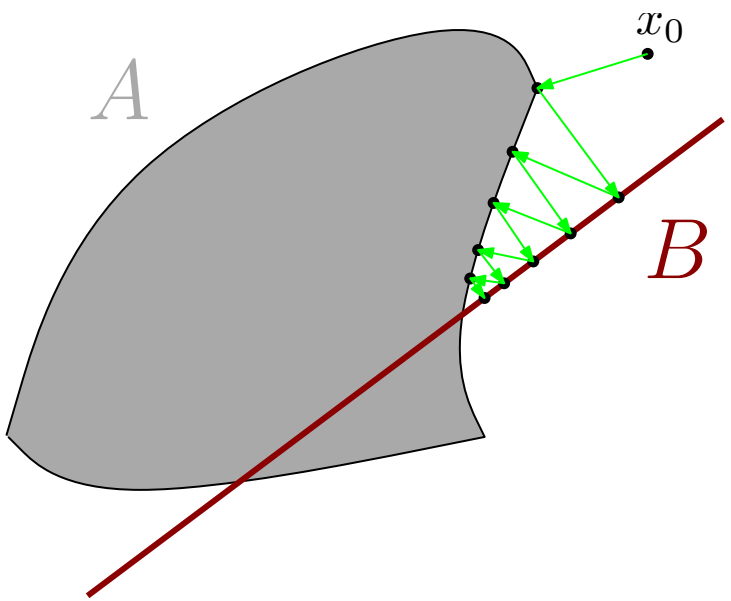
Local convergence (D-Ioffe-Lewis '13):

A and B transverse at $\bar{x}$ $\implies$ local **R**-linear convergence.

Transversality & error bounds

Coupling function:

$$\boxed{\psi(x, y) = \delta_A(x) + |x - y| + \delta_B(y).}$$



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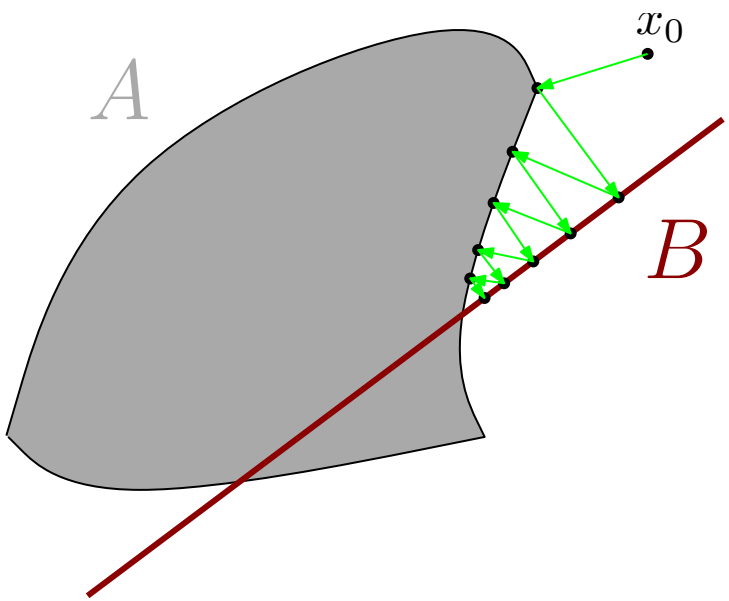
$$\Downarrow$$

$$\boxed{\max\{|\nabla\psi_x|(y),\, |\nabla\psi_y|(x)\} \geq \kappa}$$

for $x \in A$ and $y \in B$, not in $A \cap B$.



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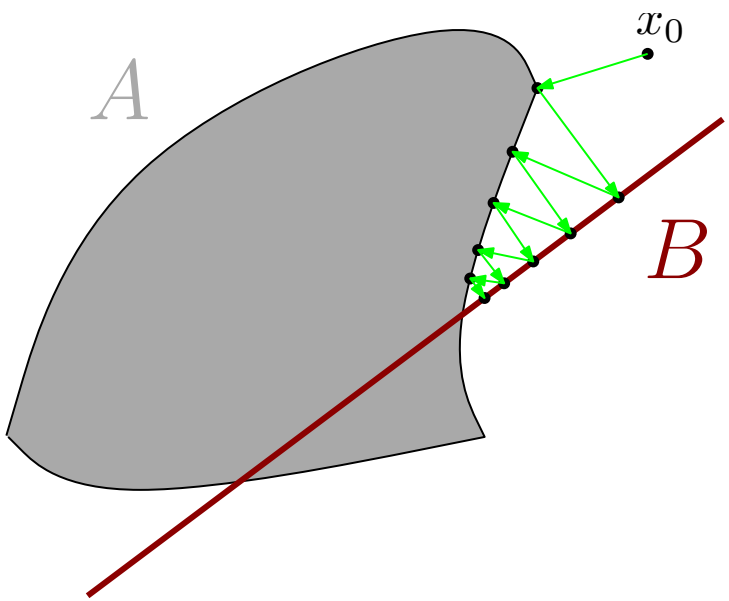

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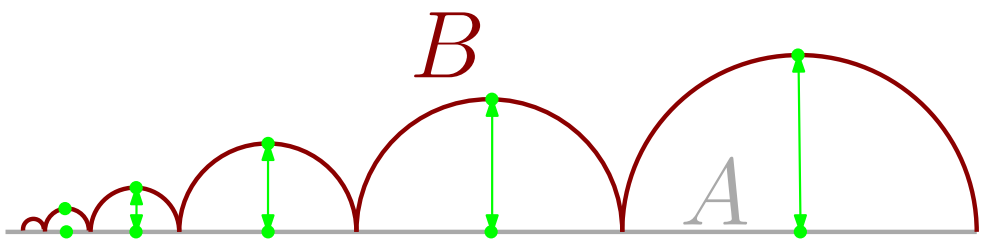

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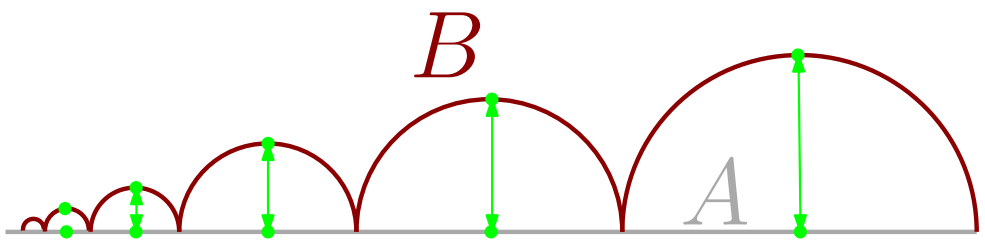

What about sublinear convergence?

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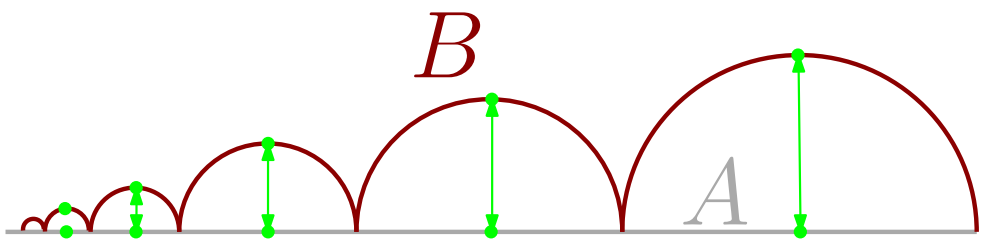

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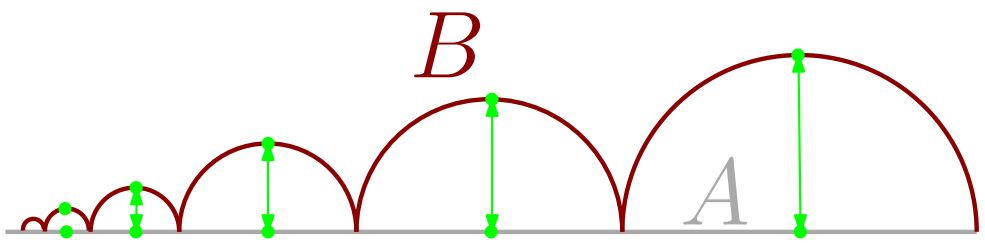

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Generic transversality (D-Ioffe-Lewis):

If A and B are semi-algebraic, then

$A + a$ and $B + b$ are transverse for a.e. (a, b)

Open question

General paradigm:

$$\text{no convexity} \quad \Longrightarrow \quad \text{no global convergence.}$$

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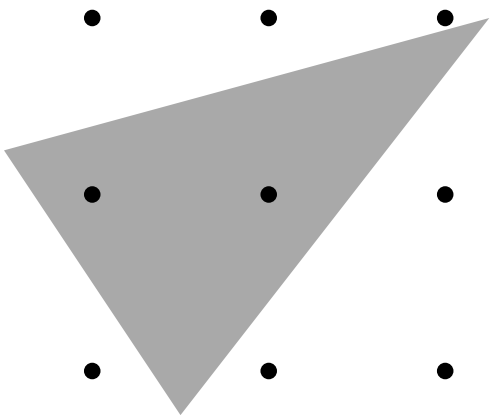
Variants of alternating projections work globally!

- Integer programming:

$$\mathbb{Z}^n \cap \{x : Ax \leq b\}$$

(eg: sudoku, 3-SAT, 4 queens problem, etc ...)



Ongoing work Artacho, Borwein, Tam.

Steepest descent curves

Steepest descent curves

Bounded speed: A curve $x\colon [0, T] \to \mathcal{X}$ is 1-Lipschitz if

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Definition (Near-steepest descent curves)

Curve $x\colon [0, T] \to \mathcal{X}$ is a near-steepest descent curve if

- x is 1-Lipschitz,
- $f \circ x$ is decreasing,
- $|\nabla(f \circ x)| \geq \overline{|\nabla f|}(x), \quad$ a.e. on $[0, T]$.

Example

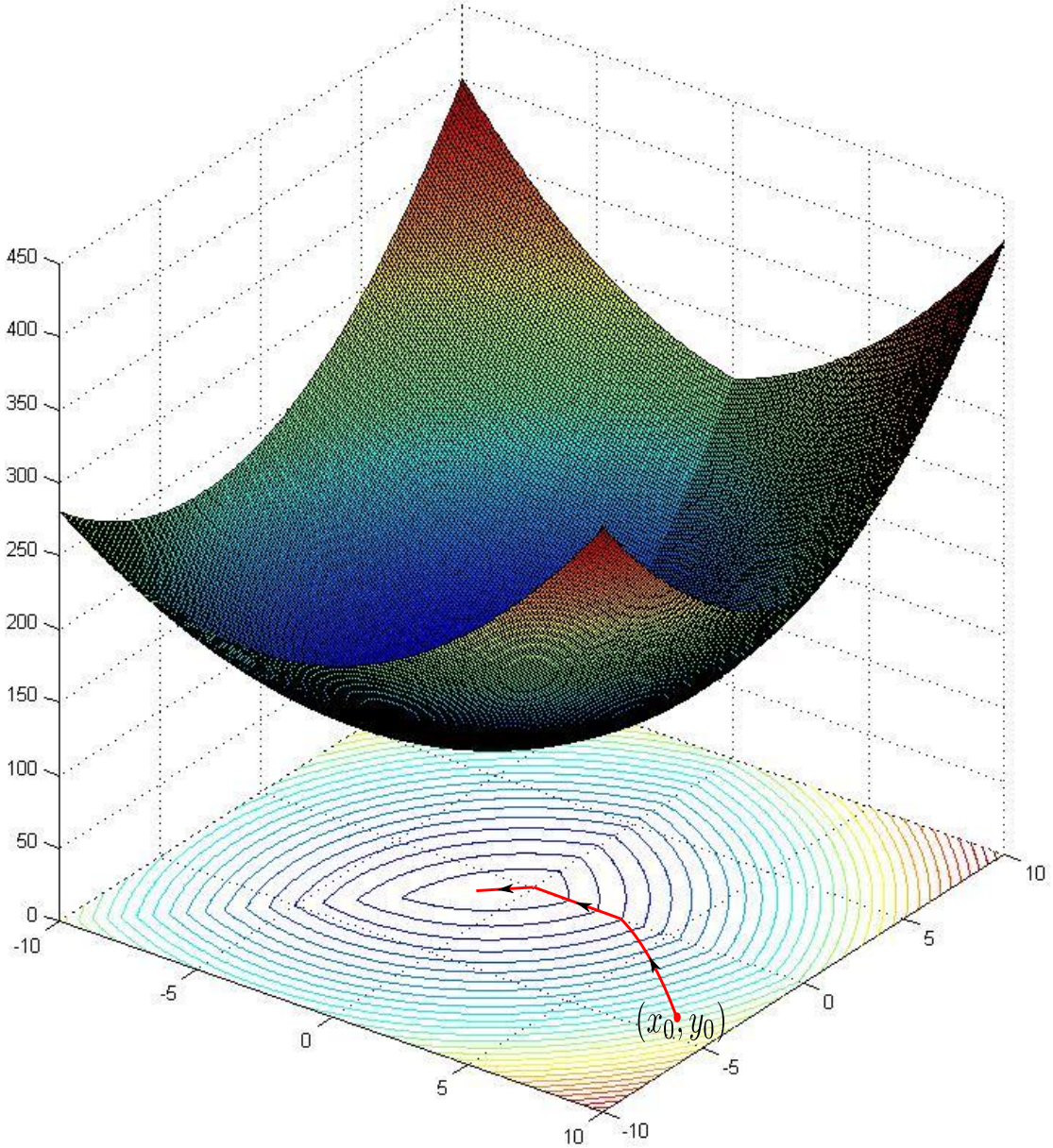

Figure: $f(x, y) = \max\{x + y, |x - y|\} + x(x + 1) + y(y + 1) + 100$

Subgradient dynamical systems

Theorem (D-Ioffe-Lewis)

For reasonable $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$ *and a curve* $x\colon [0, T] \to \mathbf{R}^n$ *the following are equivalent.*

1. x *is a near-steepest descent curve,*
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Reasonable conditions: f is smooth, convex, or semi-algebraic.

Existence

Theorem (Ambrosio et al. '05, De Giorgi '93)

For reasonable $f\colon \mathcal{X} \to \overline{\mathbf{R}}$, there exist near-steepest descent curves starting from any point.

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Proof ingredients:

- Moreau-Yosida approximation:

$$x_{k+1} = \operatorname*{argmin}_{x \in \mathcal{X}} \{f(x) + \frac{1}{2\tau} d^2(x, x_k)\}.$$

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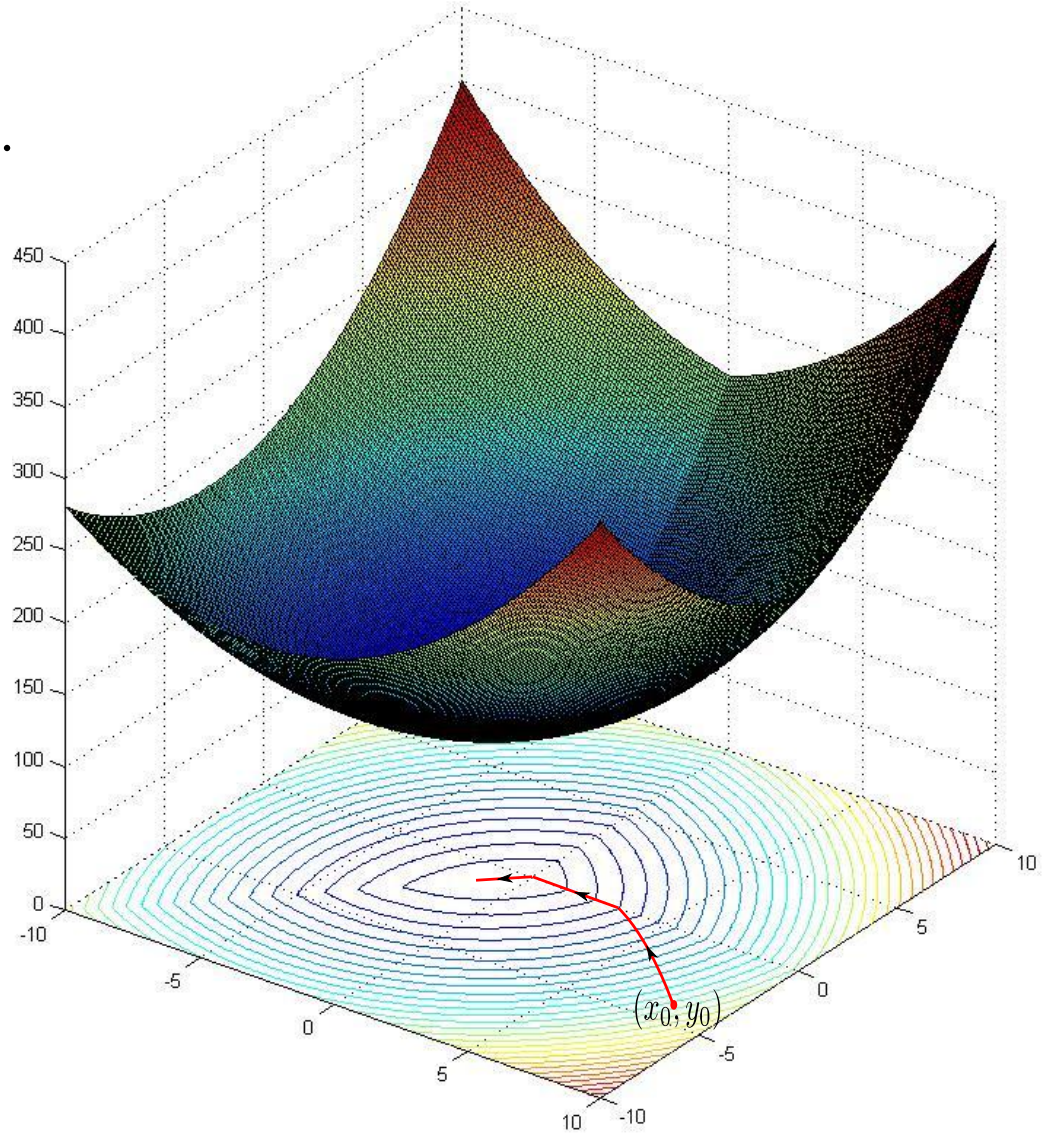
Proof is opaque and uses heavy machinery!

New proof idea: Error bound lemma

Equipartition: For $\eta > 0$,

$$f(x_0) - \eta = \tau_0 < \ldots < \tau_k = f(x_0).$$



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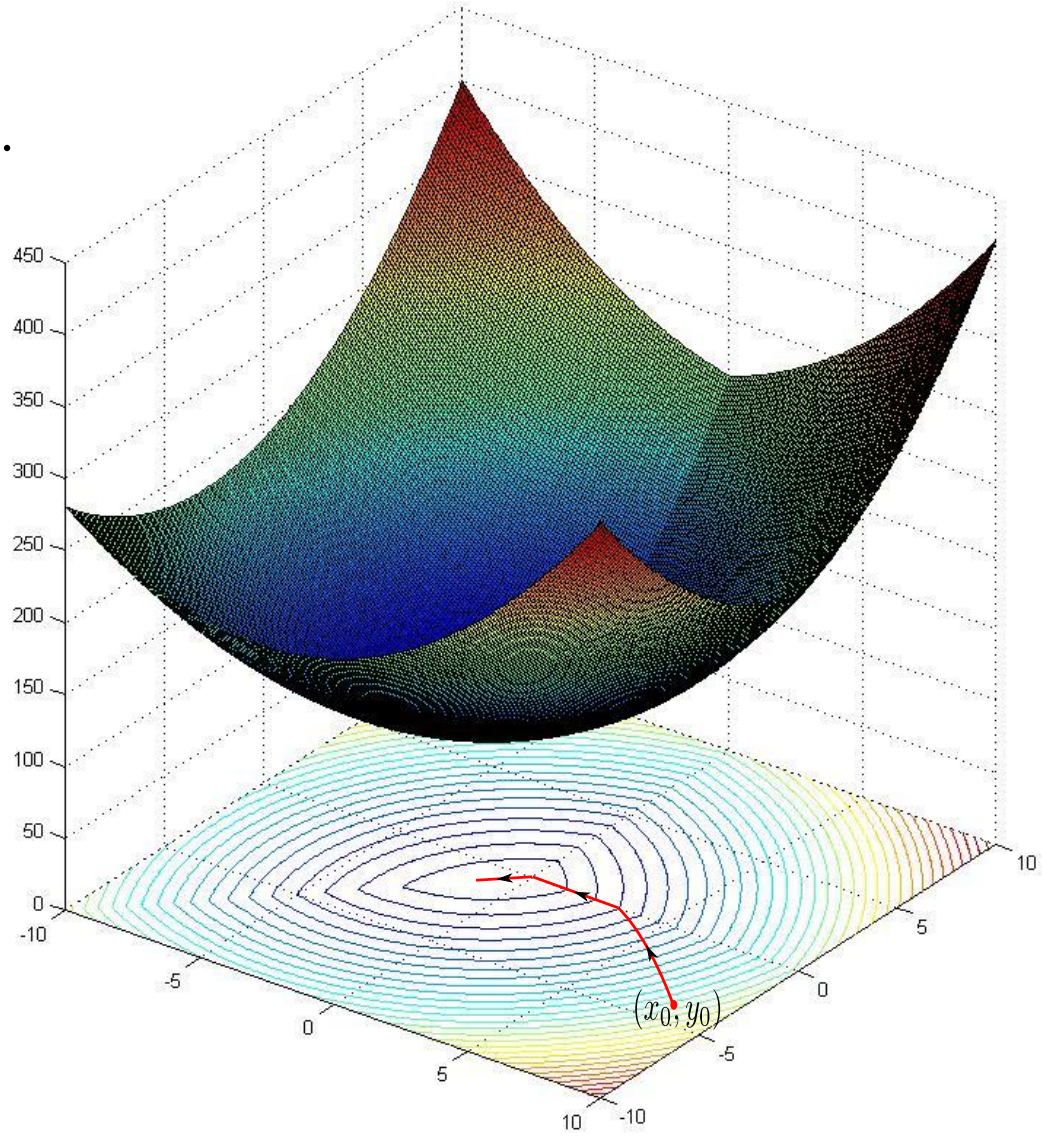

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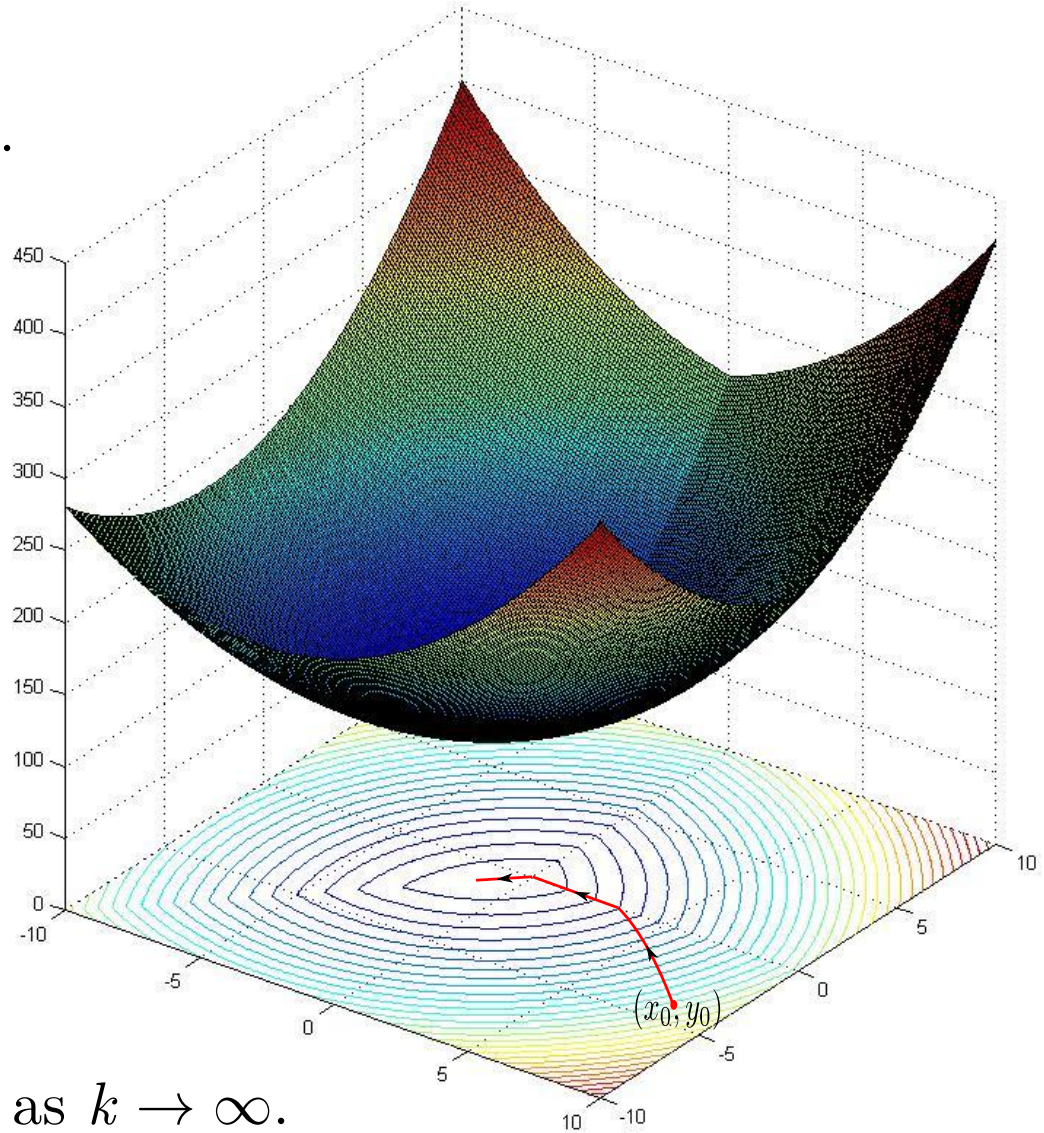

Consider resulting trajectories as $k \to \infty$.

Semi-algebraic descent

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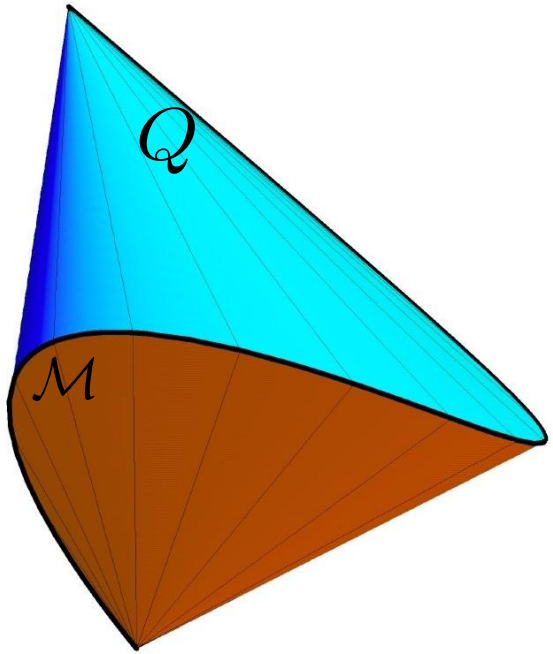
One motivation: Algorithm complexity (Eg: Attouch et al.)

Active sets in optimization.

Active sets in optimization

Figure: Q is 4×4 Toeplitz spectrahedron



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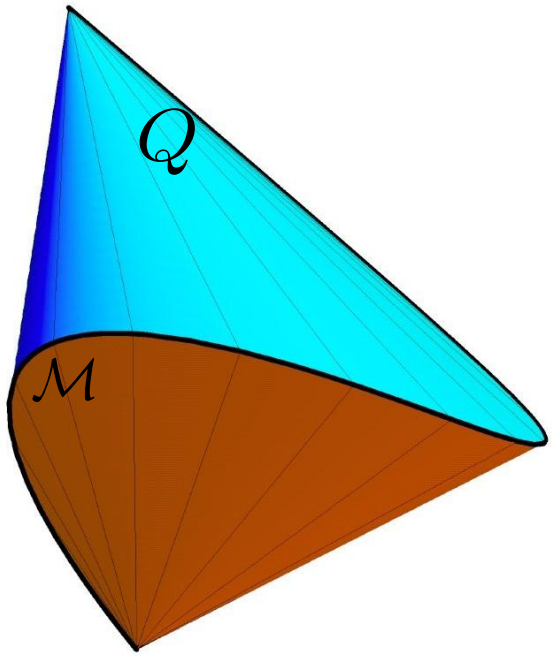

Definition (Partial Smoothness)

A set Q is partly smooth relative to $\mathcal{M} \subset Q$ if

1. **(Smoothness)** $\mathcal{M}$ is a smooth manifold,
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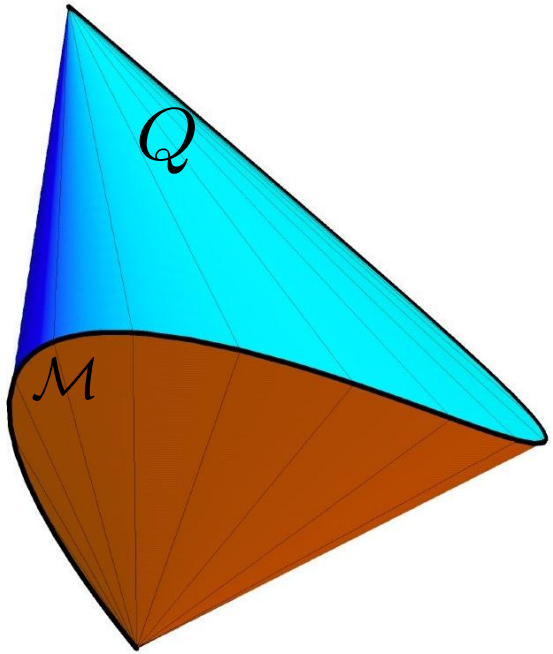

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(Originates in Lewis '03)

Active sets in optimization

Partial smoothness has classical roots!

Eg: Smooth constraints

$$Q := \{x : g_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m\}$$

where $g_1, \ldots, g_m$ are smooth.

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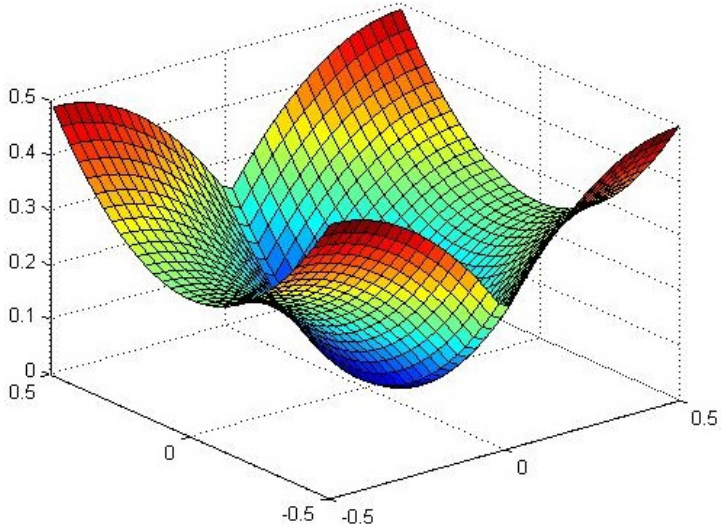

Figure: $Q = \{(x, y, z) : z \geq x(1 - x) + y^2,\ z \geq -x(1 + x) + y^2\}$

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Eg: Sum of perturbed norms

$$\min_x \; f(x) := \sum_{i=1}^{m} \|F_i(x)\|$$

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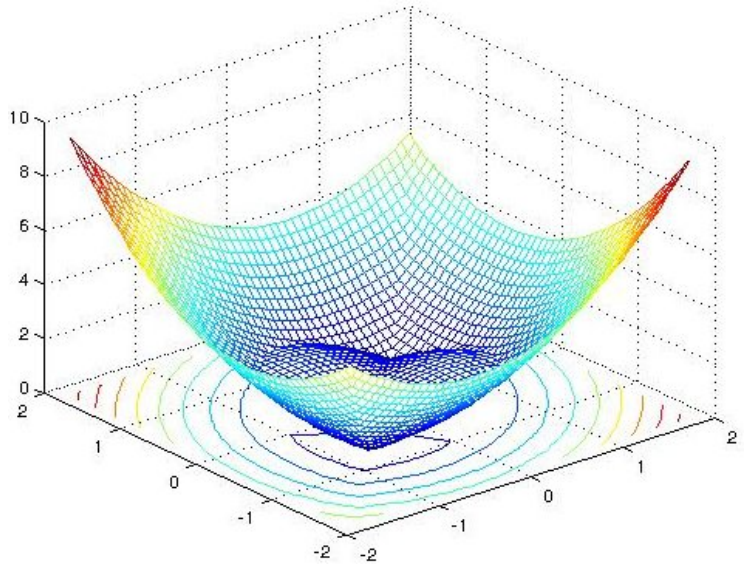

Figure: $f(x, y) := |x^2 + y^2 - 1| + |x - y|$

Active sets in optimization

Why do optimizers care?

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Then for "typical" v at any local minimizer: unique partly smooth manifold, strict complementarity, quadratic growth.

Active sets in optimization

$$\boxed{\text{Why do optimizers care?}}$$

- Many optimization algorithms identify $\mathcal{M}$ in finite time!

Eg: Gradient projection, Newton-like, proximal point.

$\Longrightarrow$ Acceleration strategies!

Eg: Wright, Burke-Moré, Ferris, Flåm, ...

Finite Identification: For $\bar{x} \in \mathcal{M}$ and $\bar{v} \in \operatorname{ri} N_Q(\bar{x})$, have

$$\boxed{\left.\begin{array}{c} x_i \to \bar{x},\, v_i \to \bar{v} \\ v_i \in N_Q(x_i) \end{array}\right\} \Longrightarrow x_i \in \mathcal{M} \text{ for all large } i}$$

finite identification $\Longleftrightarrow$ partial smoothness (D-Lewis '13)

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- cf. Spingarn-Rockafellar '79, Pataki-Tunçel '01,...

Summary

- Slope is an elegant tool.
- Variational analysis is especially effective for semi-algebraic function.
- Consequences for alternating projections, steepest descent, active sets.

Thank you.

Spectral sets

How to see this structure in eigenvalue optimization?

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Consider $\mathbf{S}^n := \{n \times n \text{ symmetric matrices}\}$ and the eigenvalue map

$$A \mapsto (\lambda_1(A), \ldots, \lambda_n(A))$$

where

$$\lambda_1(A) \leq \ldots \leq \lambda_n(A).$$

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- Partial smoothness of $\lambda^{-1}(Q)$ / partial smoothness of Q.

Spectral sets

Recognizing partial smoothness (Daniilidis-D-Lewis):

Q partly smooth at $\lambda(\bar{X})$ relative to $\mathcal{M}$

$\Longrightarrow \lambda^{-1}(Q)$ partly smooth at $\bar{X}$ relative to $\lambda^{-1}(\mathcal{M})$.

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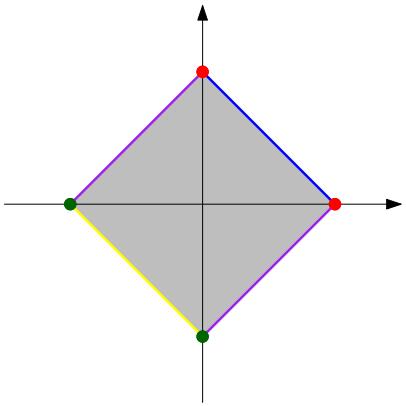
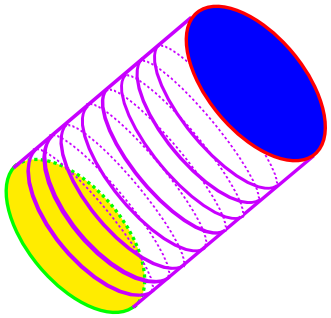
Eg:



$|x| + |y| \leq 1$



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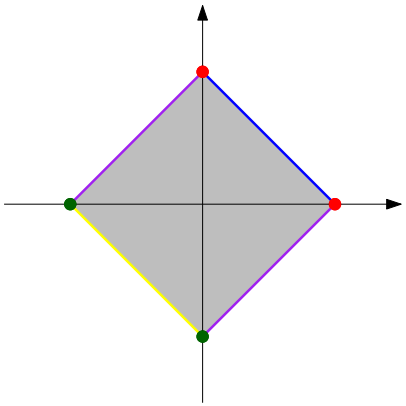

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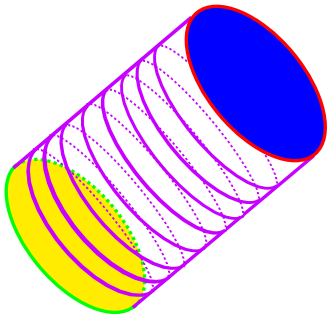

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Related work: Daniilidis-Malick-Sendov '11