Tame variational analysis

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Joint work with
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Theme:

Semi-algebraic geometry is a powerful addition to the Variational Analysis toolkit.

Illustration

For closed, convex $f\colon \mathbf{R}^n \to \overline{\mathbf{R}}$, the following are equivalent

Quadratic growth:

$$f(x) \geq f(\bar{x}) + \frac{\alpha}{2}|x - \bar{x}|^2 \qquad \text{for } x \text{ near } \bar{x}.$$

Error bound:

$$|x - \bar{x}| \leq \kappa \cdot \operatorname{dist}\big(0, \partial f(x)\big) \qquad \text{for } x \text{ near } \bar{x}.$$

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(More in Ioffe's talk tomorrow.)

Outline

- Basics of semi-algebraic geometry

- Consequences for Variational Analysis:
  - Subgradient descent
  - Sweeping process
  - Sard theorem
  - Size of subdifferential graphs
  - Approximation on singular domains

Semi-algebraic geometry

Semi-algebraic set: finite union of sets

$$\left\{ x : \quad \begin{array}{l} p_i(x) < 0 \text{ for } i \in I \\ p_j(x) = 0 \text{ for } j \in J \end{array} \right\}$$

where p_i, p_j are polynomials.

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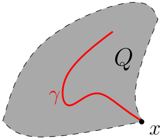
Conclusion: ∂f, $|\nabla f|$, $\operatorname{sur} F$, $\operatorname{Lip} F$, ... remain semi-algebraic.

Basic properties

Curve selection: Given $x \in \operatorname{cl} Q$, there is an analytic curve γ with $\gamma(0) = x$ and $\gamma(0, \eta) \subset Q$. (Bruhat-Cartan '50, Milnor '68)



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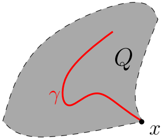

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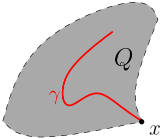

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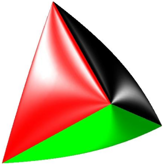
Stratification: Semialgebraic sets "stratify" into finitely many manifolds $\{\mathcal{M}_i\}$, and so have dimension.

(Whitney '65, Łojasiewicz '71)



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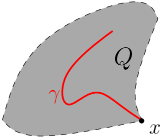

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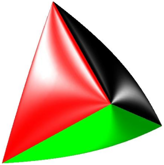

Łojasiewicz inequality: If f is semi-algebraic, then on compacta

$$\operatorname{dist}(x; f^{-1}(0)) \leq C|f(x)|^{\alpha}.$$

(Łojasiewicz '91, Kurdyka '98, Bolte-Daniilidis-Lewis '06)

What are consequences for Variational Analysis?

Subgradient systems

Theorem (D-Ioffe-Lewis)

f semi-algebraic, $\bar{x}$ not a local minimizer $\Longrightarrow$ there exists a nontrivial solution to

$$\boxed{\dot{x} \in -\partial f(x) \qquad \text{and} \qquad x(0) = \bar{x}.}$$

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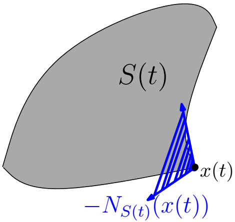
Many analogues for descent methods; e.g. proximal point, splitting, Gauss-Seidel, etc (Attouch, Bolte, Bot, Noll, Peypouquet, Soubeyran, Svaiter, ...)

Sweeping process (Moreau '77):

$$\dot{x}(t) \in -N_{S(t)}(x(t))$$

with $S(t)$ a moving set.



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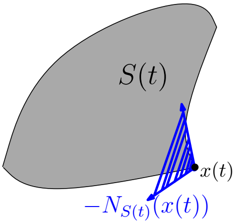

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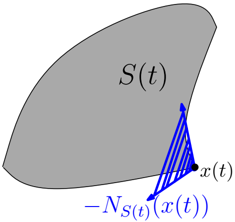

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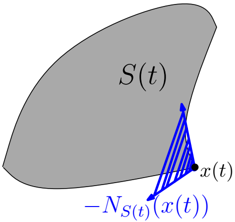

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Key estimate:

$$|\dot{x}(t)| \leq \operatorname{Lip} S(t|x(t)) \leq \sup_{x \in S(t) \cap X} \operatorname{Lip} S(t|x)$$

and the upper-bound is integrable by the Łojasiewicz inequality.

Sard Theorem

Mapping $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ is metrically regular at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if

$$\frac{\operatorname{dist}(x, F^{-1}(y))}{\operatorname{dist}(y, F(x))} \quad \text{is bounded} \quad \text{near } (\bar{x}, \bar{y}).$$

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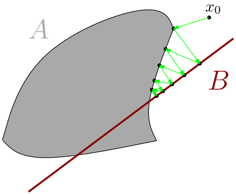
Theorem (Ioffe '08)

Semi-algebraic $F\colon \mathbf{R}^n \rightrightarrows \mathbf{R}^m$ have almost no critical values.

- Justifies typical linear convergence of basic schemes, e.g. alternating projections (D-Ioffe-Lewis '13).



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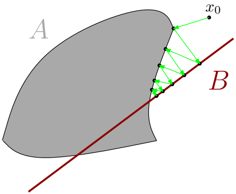

Sard Theorem & "$\operatorname{gph} \partial f$ is thin"

$\Longrightarrow$ generic properties of semi-algebraic functions.

(cf. Lewis' talk)

Generic properties

Consider

$$\boxed{\min_x f(x) + h(G(x) + y) - \langle v, x \rangle}$$

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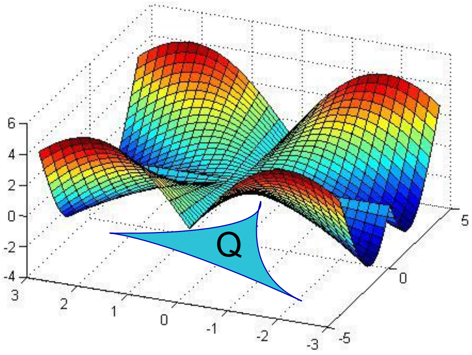
Remark: Without semi-algebraicity, one needs geometric measure theory and not all properties above are generic.

Approximation of functions

Set-up: $Q \hookrightarrow \mathbf{R}^n \xrightarrow{f} \mathbf{R}.$



Assume Q is a disjoint union of manifolds

$$\boxed{Q = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \ldots \cup \mathcal{M}_{k-1} \cup \mathcal{M}_k}$$

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Theorem (D-Larsson)

Given a continuous $f\colon \mathbf{R}^n \to \mathbf{R}$ *and any* $\epsilon > 0$, *there exists a* $\mathbf{C}^1$*-smooth* $\tilde{f}$ *satisfying*

1. **Closeness:** $|\tilde{f}(x) - f(x)| < \epsilon$ *for all* $x \in \mathbf{R}^n$,
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- For semi-algebraic Q, Whitney stratifications always exist!

Conclusion

- Semi-algebraic geometry is a powerful addition to the Variational Analysis toolkit.

- **Applications:** quadratic growth and error bounds, subgradient descent and the sweeping process, Sard theorem, and approximation on singular domains.

Thank you.

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