Optimization and intrinsic geometry

Dmitriy Drusvyatskiy
School of ORIE, Cornell University

Joint work with
A. Daniilidis (Chile), A.D. Ioffe (Technion), A.S. Lewis (Cornell)

July 31, 2013

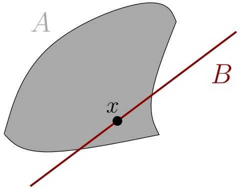
Outline

- **Method of alternating projections**

- **Active sets in optimization**

Method of alternating projections

Given sets $A, B \subset \mathbf{R}^n$, find some point $x \in A \cap B$.



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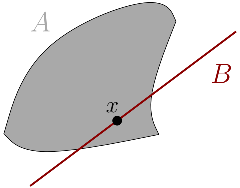

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$$d_B(x) = \min_{y \in B} |x - y| \quad \text{and} \quad P_B(x) = \{\text{nearest points of } B \text{ to } x\}.$$

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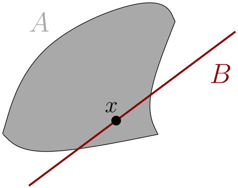

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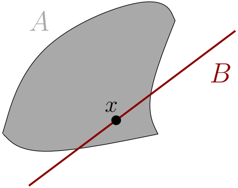

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Method of alternating projections (von Neumann '33):

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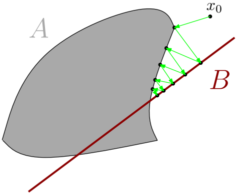

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$P_{\{X \succeq 0:\, \operatorname{rank} X \leq r\}}(Y) \Longleftrightarrow$ diagonalize, set $n - r$ smallest eigenvalues to zero, set negative eigenvalues to zero.

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Versatile:

- Inverse eigenvalue problems (Chen, Chu '96)
- Pole placement (Orsi, Yang '06)
- Information theory (Tropp, Dhillon, Heath, Strohmer '05)
- Low-order control design (Grigoriadis, Skelton '96)
- Image processing (Bauschke, Combettes, Luke '02)
- Hubble telescope (NASA '95)

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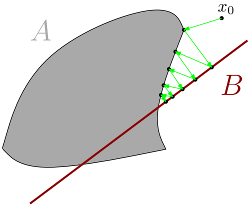

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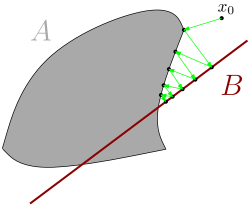

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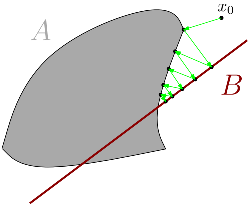

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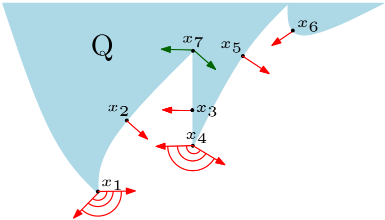

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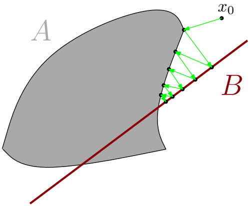

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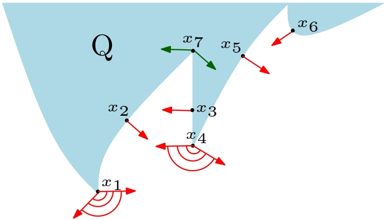

Convergence theory: Bauschke, Borwein, Combettes, Deutsch, Lewis, Luke, Malick, Phan, Trussell, von Neumann, Wang etc. . .

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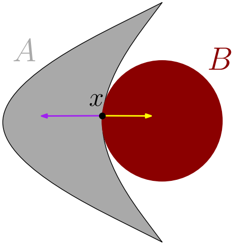

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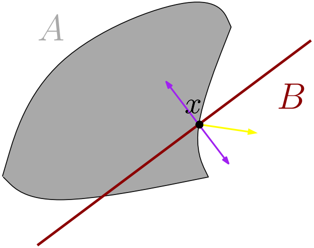

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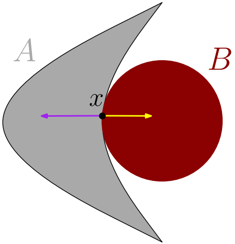

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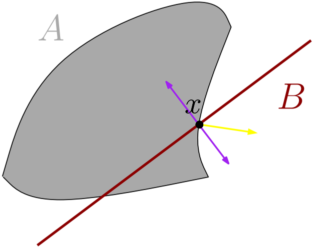

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Convergence of alternating projections (D-Ioffe-Lewis '13):

A and B transverse at $\bar{x}$ $\implies$ local **R**-linear convergence.

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Lemma (Error bound)

The following are equivalent.

Non-criticality:

$$|\nabla f| \ge \frac{1}{\kappa}$$

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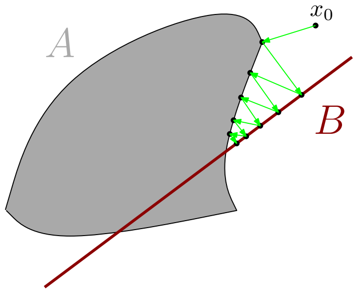
- Observed by Azé-Corvellec '04, Ioffe '00.

Coupling function:

$$\psi(x, y) = \delta_A(x) + |x - y| + \delta_B(y).$$



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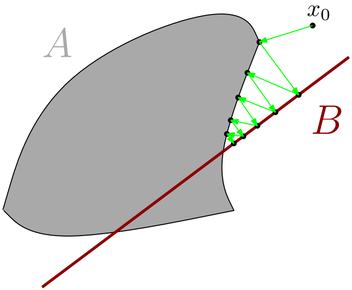

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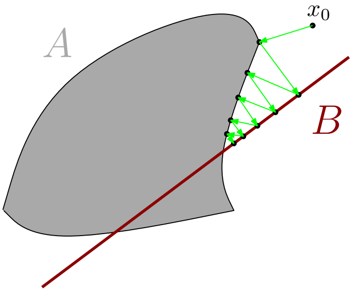
Local linear convergence



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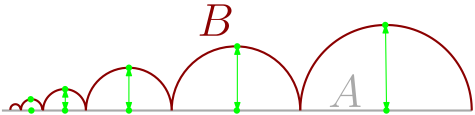

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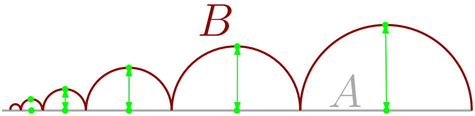

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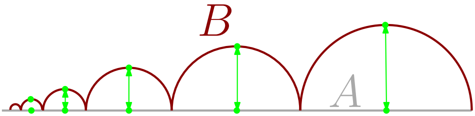

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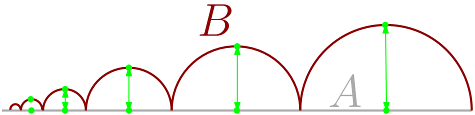

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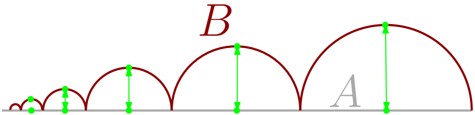

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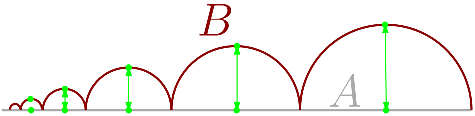

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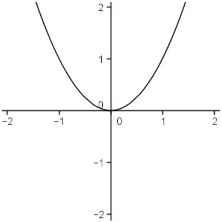
Generic transversality (D-Ioffe-Lewis):

If A and B are semi-algebraic, then

$A + a$ and $B + b$ are transverse for a.e. (a, b)

Why semi-algebraicity?



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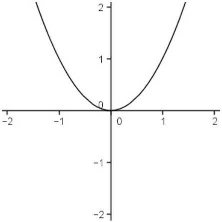

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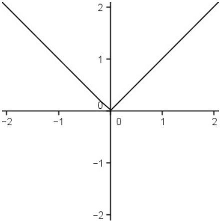

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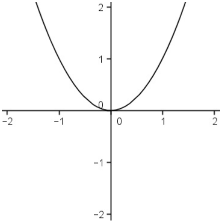

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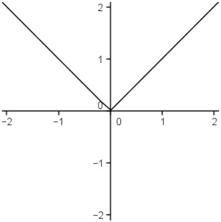

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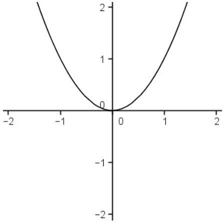

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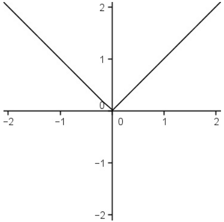

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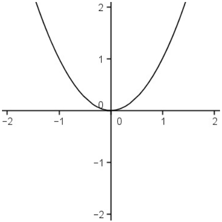

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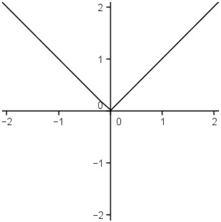

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- Apply to

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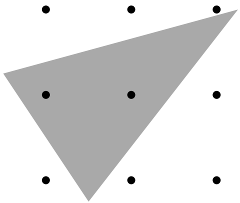
Variants of alternating projections work globally!

- Integer programming:

$$\mathbb{Z}^2 \cap \{x : Ax \leq b\}$$

(eg: sudoku, 3-SAT, 4 queens problem, etc . . .)



Ongoing work with Artacho, Borwein.

Active sets in optimization.

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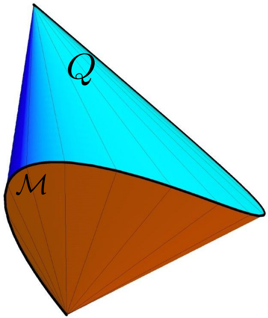
Figure: Q is 4×4 Toeplitz spectrahedron



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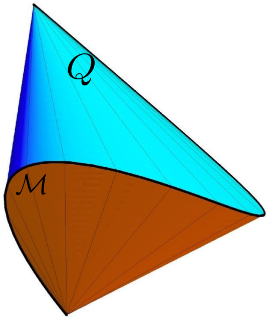


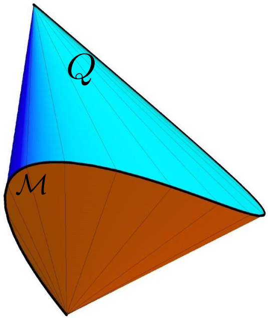
Definition (Partial Smoothness)

A set Q is partly smooth relative to $\mathcal{M} \subset Q$ if

1. **(Smoothness)** $\mathcal{M}$ is a smooth manifold,
2. **(Sharpness)** $N_{\mathcal{M}} = \operatorname{span} N_Q$ on $\mathcal{M}$,
3. **(Continuity)** N_Q varies continuously on $\mathcal{M}$.

Active sets in optimization

Figure: Q is 4×4 Toeplitz spectrahedron



Definition (Partial Smoothness)

A set Q is partly smooth relative to $\mathcal{M} \subset Q$ if

1. **(Smoothness)** $\mathcal{M}$ is a smooth manifold,
2. **(Sharpness)** $N_{\mathcal{M}} = \operatorname{span} N_Q$ on $\mathcal{M}$,
3. **(Continuity)** N_Q varies continuously on $\mathcal{M}$.

(Originates in Lewis '03)

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Partial smoothness has classical roots!

Eg: Smooth constraints

$$Q := \{x : g_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m\}$$

where $g_1, \ldots, g_m$ are smooth.

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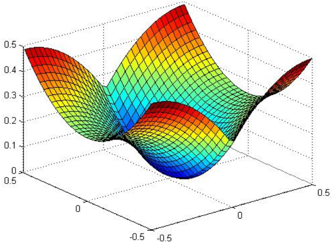

Figure: $Q = \{(x, y, z) : z \ge x(1-x) + y^2,\ z \ge -x(1+x) + y^2\}$

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Eg: **Sum of perturbed norms**

$$\min_x \; f(x) := \sum_{i=1}^{m} \|F_i(x)\|$$

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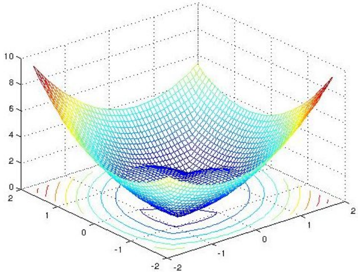

Figure: $f(x, y) := |x^2 + y^2 - 1| + |x - y|$

Why do optimizers care?

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- Many optimization algorithms identify $\mathcal{M}$ in finite time!

Eg: Gradient projection, Newton-like, proximal point.

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Finite Identification: For $\bar{x} \in \mathcal{M}$ and $\bar{v} \in \operatorname{ri} N_Q(\bar{x})$, have

$$\left.\begin{array}{c} x_i \to \bar{x},\, v_i \to \bar{v} \\ v_i \in N_Q(x_i) \end{array}\right\} \Longrightarrow x_i \in \mathcal{M} \text{ for all large } i$$

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How to see this structure in eigenvalue optimization?

Spectral sets

Consider $\mathbf{S}^n := \{n \times n \text{ symmetric matrices}\}$ and the eigenvalue map

$$X \mapsto (\lambda_1(X), \ldots, \lambda_n(X))$$

where

$$\lambda_1(X) \leq \ldots \leq \lambda_n(X).$$

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How to describe partly smooth structure of $\lambda^{-1}(Q)$?

Spectral sets

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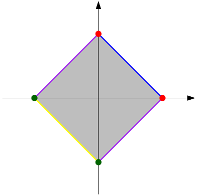

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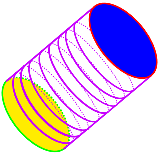

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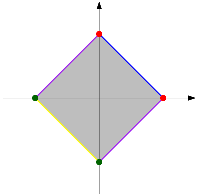

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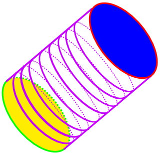

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Recognizing partial smoothness (Daniilidis-D-Lewis):

Q partly smooth at $\lambda(\bar{X})$ relative to $\mathcal{M}$

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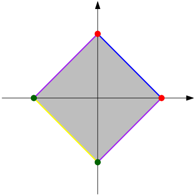

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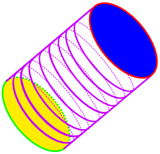

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For Q convex polyhedral cone,

$$\lambda^{-1}(Q) \quad \overset{*}{\longleftrightarrow} \quad [\lambda^{-1}(Q)]^*$$

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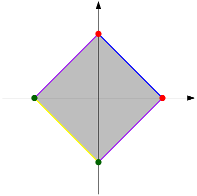

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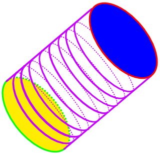

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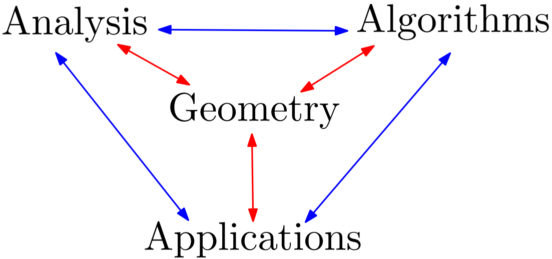
(Matrix-valued Bessel processes (D-Larsson '13))


Analysis
Algorithms
Geometry
Applications

Thank you.