Suppose $a = a(i,j)$ and $b = b(i,j)$ are two planes of indeterminants. Then for each $n \geq 0$, rational functions $R_n(i,j)$ are defined as follows:

$$\begin{aligned} R_0(i,j;a,b) &= a(i,j) \\ R_1(i,j;a,b) &= b(i,j) \end{aligned}$$

For $n \geq 2$, the $R_n$ are defined recursively by Sylvester's Formula (SF):

$$R_{n+1}(i,j;a,b) = \frac{R_n(i+1,j+1) \cdot R_n(i,j) - R_n(i,j+1) \cdot R_n(i+1,j)}{R_{n-1}(i+1,j+1)}$$

To indicate the dependence of $R_n(\ldots)$ on the level $n$, the position (i,j), and the variables $a(i,j)$ and $b(i,j)$, the notation should be $R_n(i,j;a,b)$. The variables $a(i,j)$ and $b(i,j)$ may sometimes be suppressed. Sometimes the indices $(i,j)$ may be suppressed also. The words "in a and b" stands for "in the indeterminants $a(i,j)$ and $b(i,j)$". We want to show that

**Lemma 1** Each $R_n(i,j;a,b)$ is a Laurent polynomial in a and b. Specifically, $R_n(i,j;a,b)$ can be expressed as:

$$R_n(i,j;a,b) = \frac{P_n(i,j;a,b)}{Q_n(i,j;a,b)}$$

where $P_n(i,j;a,b)$ is a polynomial in a and b and $Q_n(i,j;a,b)$ is a monomial in a and b. $P_n(i,j;a,b)$ and $Q_n(i,j;a,b)$ are assumed to have no common factors.

**Proof:** By induction on n. Nothing to prove for $R_0$ or $R_1$. For $n = 2$, (SF) shows

$$c(i,j) = R_2(i,j) = \frac{b(i+1,j+1) \cdot b(i,j) - b(i,j+1) \cdot b(i+1,j)}{a(i+1,j+1)}$$

For $n = 3$, a direct computation shows that $R_3(\ldots) = \dfrac{P_3(\ldots)}{Q_3(\ldots)}$ where:

$$Q_3(i,j) = a(i+1,j+1) \cdot a(i+2,j+2) \cdot a(i+1,j+2) \cdot a(i+2,j+1) \cdot b(i+1,j+1)$$

$$\begin{aligned} (i) \qquad R_{n+1}(i,j;a,b)) &= \frac{N_{n+1}}{R_{n-1}(i+1,j+1;a,b)} \\ (ii) \qquad R_{n+1}(i,j;a,b) &= R_n(i,j;b,c) \end{aligned}$$

Setting (i) equal to (ii), we have

$$\frac{N_{n+1}}{R_{n-1}(i,j;a,b)} = \frac{P_n(i,j;b,c)}{Q_n(i,j;b,c)}$$

In (i) the numerator for the expression for $R_{n+1}$ is obtained by (SF) as:

$$N_{n+1} = R_n(i+1, j+1) \cdot R_n(i,j) - R_n(i, j+1) \cdot R_n(i+1, j)$$

Each of these $R_n(\ldots)$ is a Laurent polynomial in a and b whose denominator is a product of a's and b's. Combing terms, $N_{n+1}$ has the form

$$N_{n+1}(i,j;a,b) = \frac{S(a,b)}{T(a,b)}$$

where T(a,b) is a product of a's and b's. We also have

$$R_{n-1}(i,j;a,b) = \frac{P_{n-1}(i,j;a,b)}{Q_{n-1}(i,j;a,b)}$$

Therefore [eqn 3]:

$$S(a,b) \cdot Q_n(i,j;b,c) \cdot Q_{n-1}(i,j;a,b) = P_n(i,j;b,c) \cdot P_{n-1}(i,j;a,b) \cdot T(a,b)$$

By the inductive assumption, $Q_{n-1}(i,j;a,b)$ is a monomial in the a's and b's. Also $Q_n(i,j;b,c)$ is a monomial in the b's and c's. Using the expression for the c's, and multiplying by $M(a)$ (a monomial in the a's) we may write

$$Q_n(i,j;b,c) \cdot M(a) = U(b) \cdot V(b)$$

where U(b) is a monomial in the b's, and V(b) is a product of expressions of the form

$$v(h,k) = b(h,k) \cdot b(h+1,k+1) - b(h+,k) \cdot b(h,k+1)$$

After multiplying by $M(a)$, eqn 3 becomes

$$S(a,b) \cdot U(b) \cdot V(b) \cdot Q_{n-1}(i,j;a,b) = P_n(i,j;b,c) \cdot P_{n-1}(i,j;a,b) \cdot T(a,b)$$

$U(b) \cdot V(b) \cdot Q_{n-1}(i,j;a,b)$ and $P_{n-1}(i,j;a,b)$ are relatively prime (see Remark). Therefore $P_{n-1}(i,j;a,b)$ must divide $S(a,b)$. Hence $R_{n+1}(i,j;a,b)$ has the desired form as a Rational function with monomial denominators.

**Remark:** If all the $a(i,j)$ are specialized to 1, and the b's are specialized so that $b(j,i) = b(i,j)$ each of the $v(h,k)$ is a Hankel det in the b's. $P_{n-1}(i,j;1,b)$ is also a Hankel det in the b's, necessarily different from the v's (because degree 3 or higher). Therefore $P_{n-1}(i,j;a,b)$ is relatively prime to

$$U(b) \cdot V(b) \cdot Q_{n-1}(i,j;a,b)$$

$A = \{a_{i,j}\}, \quad i, j \geq 0$

$B = \{b_{i,j}\}, \quad i, j \geq 0$

$C = \{c_{i,j}\}, \quad i, j \geq 0, \quad \text{where} \quad a_{i+1,j+1} \cdot c_{i,j} = b_{i,j} b_{i+1,j+1} - b_{i,j+1} b_{i+1,j}$

$D = \{d_{i,j}\}, \quad i, j \geq 0, \quad \text{where} \quad b_{i+1,j+1} \cdot d_{i,j} = c_{i,j} c_{i+1,j+1} - c_{i,j+1} c_{i+1,j}$

$E = \{e_{i,j}\}, \quad i, j \geq 0, \quad \text{where} \quad c_{i+1,j+1} \cdot e_{i,j} = d_{i,j} d_{i+1,j+1} - d_{i,j+1} d_{i+1,j}$

$F = \ldots, \quad i, j \geq 0$ etc

Hales' Identity is: $e_{0,0} b_{1,1} b_{2,2} + a_{2,2} d_{0,1} d_{1,0} = \det \mathcal{C}, \quad \text{where} \quad \mathcal{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} \\ c_{1,0} & c_{1,1} & c_{1,2} \\ c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}$

To establish this formula, start with:

$$e_{0,0} c_{1,1} = d_{0,0} d_{1,1} - d_{0,1} d_{1,0}$$

$$e_{0,0} c_{1,1} b_{1,1} b_{2,2} b_{1,2} b_{2,1} = (c_{0,0} c_{1,1} - c_{0,1} c_{1,0})(c_{1,1} c_{2,2} - c_{1,2} c_{2,1}) b_{1,2} b_{2,1} - (c_{0,1} c_{1,2} - c_{0,2} c_{1,1})(c_{1,0} c_{2,1} - c_{1,1} c_{2,0}) b_{1,1} b_{2,2}$$

$$= (c_{0,0} c_{1,1} - c_{0,1} c_{1,0})(c_{1,1} c_{2,2} - c_{1,2} c_{2,1}) b_{1,2} b_{2,1} - (c_{0,1} c_{1,2} - c_{0,2} c_{1,1})(c_{1,0} c_{2,1} - c_{1,1} c_{2,0}) b_{1,2} b_{2,1} + (c_{0,1} c_{1,2} - c_{0,2} c_{1,1})(c_{1,0} c_{2,1} - c_{1,1} c_{2,0}) a_{2,2} c_{1,1}$$

$$= (c_{0,0} c_{1,1} c_{1,1} c_{2,2} - c_{0,0} c_{1,1} c_{1,2} c_{2,1} - c_{0,1} c_{1,0} c_{1,1} c_{2,2} + c_{0,1} c_{1,0} c_{1,2} c_{2,1} + {} - c_{0,1} c_{1,2} c_{1,0} c_{2,1} + c_{0,1} c_{1,2} c_{1,1} c_{2,0} + c_{0,2} c_{1,1} c_{1,0} c_{2,1} - c_{0,2} c_{1,1} c_{1,1} c_{2,0}) b_{1,2} b_{2,1} - a_{2,2} d_{0,1} d_{1,0} b_{1,2} b_{2,1} c_{1,1}$$

$$= \det \mathcal{C} \cdot b_{1,2} b_{2,1} c_{1,1} - a_{2,2} d_{0,1} d_{1,0} b_{1,2} b_{2,1} c_{1,1}$$

Dividing by $c_{1,1} b_{1,2} b_{2,1}$ gives Hales' formula.

There is a similar formula with subscripts interchanged: $e_{0,0} b_{1,2} b_{2,1} + a_{2,2} d_{0,0} d_{1,1} = \det \mathcal{C}$

There is also a similar formula with the octahedron corners $E_{0,0}$ and $A_{2,2}$ replaced by corners $C_{0,0}$ and $C_{2,2}$, and another one with corners $C_{0,2}$ and $C_{2,0}$.

Assuming an error of $\Delta$ at $C_{1,1}$ The calculation of the error at $E_{0,0}$

$$\Delta e_{0,0} = \left[\frac{c_{0,0}d_{1,1}}{b_{1,1}c_{1,1}} + \frac{c_{2,2}d_{0,0}}{b_{2,2}c_{1,1}} + \frac{c_{0,2}d_{1,0}}{b_{1,2}c_{1,1}} + \frac{c_{2,0}d_{0,1}}{b_{2,1}c_{1,1}} - \frac{e_{0,0}}{c_{1,1}}\right]\Delta \qquad \text{Let } \delta e_{0,0} = \frac{\Delta e_{0,0}}{\Delta}.$$

$$(\delta e_{0,0}) \cdot b_{1,1}b_{2,2}b_{1,2}b_{2,1}c_{1,1} = c_{0,0}d_{1,1}b_{2,2}b_{1,2}b_{2,1} + c_{2,2}d_{0,0}b_{1,1}b_{1,2}b_{2,1} +$$

$$c_{0,2}d_{1,0}b_{2,1}b_{1,1}b_{2,2} + c_{2,0}d_{0,1}b_{1,2}b_{1,1}b_{2,2} - e_{0,0}b_{1,1}b_{2,2}b_{1,2}b_{2,1}$$

$$= \left[c_{0,0}(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + c_{2,2}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0})\right] \cdot b_{1,2}b_{2,1} +$$

$$\left[c_{0,2}d_{1,0}b_{2,1} + c_{2,0}d_{0,1}b_{1,2}\right] \cdot \left[c_{1,1}a_{2,2} + b_{1,2}b_{2,1}\right] - \det\mathcal{C} \cdot b_{1,2}b_{2,1} + a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}$$

$$= \Big[c_{0,0}(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + c_{2,2}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}) +$$

$$c_{0,2}(c_{1,0}c_{2,1} - c_{1,1}c_{2,0}) + c_{2,0}(c_{0,1}c_{1,2} - c_{0,2}c_{1,1})\Big] \cdot b_{1,2}b_{2,1} - \det\mathcal{C} \cdot b_{1,2}b_{2,1} +$$

$$(c_{0,2}c_{0,2}d_{1,0}b_{2,1} + c_{2,0}d_{0,1}b_{1,2})c_{1,1}a_{2,2} + a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}$$

$$= (c_{0,0}c_{2,2} - c_{0,2}c_{2,0})b_{1,2}b_{2,1}c_{1,1} + c_{2,0}d_{0,1}b_{1,2}a_{2,2}c_{1,1} + c_{0,2}d_{1,0}b_{2,1}a_{2,2}c_{1,1} + d_{0,1}d_{1,0}a_{2,2}b_{1,2}b_{2,1}$$

Then $$\Delta e_{0,0} = \left[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{b_{1,1}b_{2,2}} + \frac{c_{2,0}d_{0,1}a_{2,2}}{b_{2,1}b_{1,1}b_{2,2}} + \frac{c_{0,2}d_{1,0}a_{2,2}}{b_{1,2}b_{1,1}b_{2,2}} + \frac{a_{2,2}d_{1,0}d_{0,1}}{c_{1,1}b_{1,1}b_{2,2}}\right] \cdot \Delta$$

Start with $\Delta c_{1,1} = \Delta = \frac{b_{1,1}b_{2,2}}{a_{2,2}} \cdot \epsilon$, then

$$\Delta e_{0,0} = \left[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{a_{2,2}} + \frac{c_{2,0}d_{0,1}}{b_{2,1}} + \frac{c_{0,2}d_{1,0}}{b_{1,2}} + \frac{d_{1,0}d_{0,1}}{c_{1,1}}\right] \cdot \epsilon$$

This guarantees that $\Delta e_{0,0}$ has an expression as a sum of terms, each of which has valuation $\geq p - q$, where $p$ is the precision and $q =$ max valuation of $a_{2,2}, b_{1,1}, b_{2,2}, b_{1,2}, b_{2,1}, c_{1,1}$

The important thing is that $\Delta e_{0,0} \cdot (b_{1,1}b_{2,2}b_{1,2}b_{2,1}c_{1,1})$ belongs to the ideal generated by

$$b_{1,2}b_{2,1}c_{1,1},\ a_{2,2}b_{1,2}c_{1,1},\ a_{2,2}b_{2,1}c_{1,1},\ a_{2,2}b_{1,2}b_{2,1},\ b_{1,1}b_{2,2}c_{1,1},\ a_{2,2}b_{1,1}c_{1,1},\ a_{2,2}b_{2,2}c_{1,1},\ a_{2,2}b_{1,1}b_{2,2}$$

There is a similar formula with denominators $b_{1,1}$ and $b_{2,2}$ instead of $b_{1,2}$ and $b_{2,1}$:

$$\Delta e_{0,0} = \left[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{a_{2,2}} - \frac{c_{0,0}d_{1,1}}{b_{1,1}} - \frac{c_{2,2}d_{1,1}}{b_{2,2}} + \frac{d_{0,0}d_{1,1}}{c_{1,1}}\right] \cdot \epsilon$$