October 25, 2010

Dear Ed and Al,

The Somos-4 algebra that was on the blackboard yesterday isn't as hard as I was making it — a bit of juggling shows that the x_i are actually coprime, and keeping track of the epsilons and their denominators isn't all that hard.

Let x_0, x_1, x_2, x_3 be indeterminates, and set $A := \mathbf{Z}[x_i^{\pm 1}]_{0 \le i < 4}$. Let $\epsilon_0, \epsilon_1, \ldots$ be further indeterminates, and define a sequence of elements of the fraction field of $A(\epsilon_i)$ by the Somos-4 rule, with errors:

$$x_n x_{n+4} = x_n^2(1 + \epsilon_n) + x_{n+1}x_{n+3}.$$

Set

$$B := A\left[\frac{\epsilon_i}{x_j}\right]$$

where i ranges over nonnegative integers and, for each i, j satisfies $i \le j \le i + 4$.

Note that A is a UFD, whereas B isn't.

Remark: The subscripts on the errors correspond to edges, and the numbering here (slightly different from yesterday) numbers an edge with the lowest subscript of any cluster variable in the two nodes connected by an edge. Moreover, in the definition of B, the denominators that occur with a given error are exactly the (five) cluster variables that appear in the nodes connected by that edge (just as in the simple Somos case). The main result here is of course that x_n lies in B; it is easy to see that the last error term needed to produce x_n is ϵ_{n-4}/x_{n-4}, but I'll ignore that detail.

Theorem 1. *With the above notation, x_n lies in B for all n.*

The proof is by induction, following the proof of the Laurent phenomonen that we talked about yesterday. Specifically, we prove a more elaborate compound assertion (leaving the base cases to the reader). For all $n \ge 4$, the claim is that

- **(0):** $x_i \in B$ for $n \le i < n + 4$.
- **(1):** x_{n+4}, R, S are all in B, where $R := (x_n x_{n+3}^2 + x_{n+2}^3)/x_{n+1}$ and $S := (x_n^2 x_{n+3} + x_{n+1}^3)/x_{n+2}$.
- **(2):** For all $i \ne j$, $n \le i, j < n + 4$,

$$(x_i, x_j) = B,$$

i.e., there are a, b in B such that $ax_i + bx_j = 1$.

Suppose this is true for n, and consider the corresponding assertions for $n + 1$. Part **(0)** is immediate.

Part **(2)** has 3 assertions that need to be proved, namely, that x_{n+5}, R', and S' are in B, where R', S' are obtained from the definitions above by incrementing every subscript. In the formula for x_{n+5}, substitute the definition for x_{n+4}, and make a common denominator, to get:

$$\begin{aligned}
x_{n+5} &= \frac{x_{n+3}^2(1+\epsilon_{n+1})+x_{n+2}x_{n+4}}{x_{n+1}} = x_{n+3}^2\frac{\epsilon_{n+1}}{x_{n+1}} + \frac{x_{n+3}^2+x_{n+2}x_{n+4}}{x_{n+1}} \\
&= \beta + \frac{x_n x_{n+3}^2 + x_{n+2}(x_{n+2}^2(1+\epsilon_n)+x_{n+1}x_{n+3}}{x_n x_{n+1}}) \\
&= \beta + \frac{x_n x_{n+3}^2 + x_{n+2}^3(1+\epsilon_n)+x_{n+1}x_{n+2}x_{n+3}}{x_n x_{n+1}})
\end{aligned}$$

where $\beta = x_{n+3}^2\epsilon_{n+1}/x_{n+1}$ is an element of B. From the induction assumption **(2)** we get $1 = ax_n + b_{n+1}$ and hence

$$\frac{1}{x_n x_{n+1}} = \frac{a}{x_{n+1}} + \frac{b}{x_n}.$$

Therefore it suffices to prove that

$$\frac{x_{n+2}^3(1+\epsilon_n)+x_{n+1}x_{n+2}x_{n+3}}{x_n} \quad \text{and} \quad \frac{x_n x_{n+3}^2 + x_{n+2}^3(1+\epsilon_n)}{x_{n+1}}$$

are in B. The first of these is immediate from factoring out x_{n+2} and then using the definition of x_{n+4} (retracing our steps). The second quotient is just $R + \epsilon_n/x_{n+1}$ which is in B.

To prove that $R' = (x_{n+1}x_{n+4}^2 + x_{n+3}^3)/x_{n+2}$ is in B, the same procedure applies: substitute for x_{n+4}, make a common denominator $x_n^2 x_{n+2}$, use the relative primeness to reduce to the separate questions of whether the numerator is divisible by x_n^2 and x_{n+2}, observe that the former question has an obvious affirmative answer, and then do some simple algebra to verifiy divisibility by x_{n+2}. The actual algebra shows that a new ϵ_i/x_j is needed, namely ϵ_n/x_{n+2}. A similar argument works for $S' = (x_{n+1}^2 x_{n+4} + x_{n+2}^3)/x_{n+3}$, where ϵ_n/x_{n+3} occurs.

Now we turn to part **(2)**. This has three nontrivial assertions, namely that $(x_{n+4}, x_{n+i}) = B$ for $i = 1, 2, 3$.

To prove the first of these, let $I := (x_{n+4}, x_{n+1})$. From

$$x_n x_{n+4} - x_{n+1}x_{n+3} = x_{n+2}^2(1+\epsilon_n)$$

it follows that $x_{n+2}^2(1+\epsilon_n)$ is in I. From $\epsilon_n = x_{n+1}(\epsilon_n/x_{n+1})$ we see that ϵ_n is in I and therefore

$$x_{n+2}^2 = x_{n+2}^2(1+\epsilon_n) - \epsilon_n x_{n+2}^2$$

is in I. From the induction assumption $(x_{n+1}, x_{n+2}) = B$, which implies that $(x_{n+1}, x_{n+2}^2) = B$, we deduce that $I = B$ as claimed. The proof that $(x_{n+4}, x_{n+3}) = B$ follows in exactly the same manner.

Finally, let $I := (x_{n+4}, x_{n+2})$. From

$$x_n x_{n+4} - x_{n+2}^2(1+\epsilon_n) = x_{n+1}x_{n+3}$$

it follows that $x_{n+1}x_{n+3}$ is in I. Since x_{n+2} is coprime to x_{n+1} and x_{n+3}, and hence their product, it follows that $I = B$.

Regards,