THE LAURENT PHENOMENON

SERGEY FOMIN AND ANDREI ZELEVINSKY

Abstract. A composition of birational maps given by Laurent polynomials need not be given by Laurent polynomials; however, sometimes—quite unexpectedly—it does. We suggest a unified treatment of this phenomenon, which covers a large class of applications. In particular, we settle in the affirmative a conjecture of D. Gale and R. Robinson on integrality of generalized Somos sequences, and prove the Laurent property for several multidimensional recurrences, confirming conjectures by J. Propp, N. Elkies, and M. Kleber.

Contents

1. Introduction

In this paper, we suggest a unified explanation for a number of instances in which certain recursively defined rational functions prove, unexpectedly, to be Laurent polynomials. We begin by presenting several instances of this *Laurent phenomenon* established in the paper.

Example 1.1. *(The cube recurrence)* Consider a 3-dimensional array

$$(y_{ijk} \,:\, (i,j,k) \in \mathcal{H})$$

whose elements satisfy the recurrence

$$y_{i,j,k} = \frac{\alpha y_{i-1,j,k}y_{i,j-1,k-1} + \beta y_{i,j-1,k}y_{i-1,j,k-1} + \gamma y_{i,j,k-1}y_{i-1,j-1,k}}{y_{i-1,j-1,k-1}}. \quad (1.1)$$

Here $\mathcal{H}$ can be any non-empty subset of $\mathbb{Z}^3$ satisfying the following conditions:

(1.2) if $(i,j,k) \in \mathcal{H}$, then $(i',j',k') \in \mathcal{H}$ whenever $i \le i', j \le j', k \le k'$;

(1.3) for any $(i',j',k') \in \mathcal{H}$, the set $\{(i,j,k) \in \mathcal{H} : i \le i', j \le j', k \le k'\}$ is finite.

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Theorem 1.2. *Let* $H_{\rm init} = \{(a,b,c) \in \mathcal{H} : (a-1,b-1,c-1) \notin \mathcal{H}\}$. *For every* $(i,j,k) \in \mathcal{H}$, *the entry* $y_{i,j,k}$ *is a Laurent polynomial with coefficients in* $\mathbb{Z}[\alpha,\beta,\gamma]$ *in the initial entries* $y_{a,b,c}$, *for* $(a,b,c) \in H_{\rm init}$.

The cube recurrence (with $\alpha = \beta = \gamma = 1$) was introduced by James Propp [10], who was also the one to conjecture Laurentness in the case when $\mathcal{H} \subset \mathbb{Z}^3$ is given by the condition $i + j + k \geq 0$; in this case $H_{\rm init}$ consists of all $(a,b,c) \in \mathcal{H}$ such that $a + b + c \in \{0,1,2\}$. Another natural choice of $\mathcal{H}$ was suggested by Michael Kleber: $\mathcal{H} = \mathbb{Z}^3_{\geq 0}$, in which case $H_{\rm init} = \{(a,b,c) \in \mathbb{Z}^3_{\geq 0} : abc = 0\}$.

Example 1.3. *(The Gale-Robinson sequence)* Let p, q, and r be distinct positive integers, let $n = p + q + r$, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_{k+n} = \frac{\alpha y_{k+p} y_{k+n-p} + \beta y_{k+q} y_{k+n-q} + \gamma y_{k+r} y_{k+n-r}}{y_k}\,. \tag{1.4}$$

David Gale and Raphael Robinson conjectured (see [7] and [8, E15]) that every term of such a sequence is an integer provided $y_0 = \cdots = y_{n-1} = 1$ and α, β, γ are positive integers. Using Theorem 1.2, we prove the following stronger statement.

Theorem 1.4. *As a function of the initial terms* $y_0, \ldots, y_{n-1}$, *every term of the Gale-Robinson sequence is a Laurent polynomial with coefficients in* $\mathbb{Z}[\alpha,\beta,\gamma]$.

We note that the special case $\alpha = \beta = \gamma = 1$, $p = 1$, $q = 2$, $r = 3$, $n = 6$ (resp., $r = 4$, $n = 7$) of the recurrence (1.4) is the Somos-6 (resp., Somos-7) recurrence [7].

Example 1.5. *(Octahedron recurrence)* Consider the 3-dimensional recurrence

$$y_{i,j,k} = \frac{\alpha y_{i+1,j,k-1} y_{i-1,j,k-1} + \beta y_{i,j+1,k-1} y_{i,j-1,k-1}}{y_{i,j,k-2}} \tag{1.5}$$

for an array $(y_{ijk})_{(i,j,k) \in \mathcal{H}}$ whose indexing set $\mathcal{H}$ is contained in the lattice

$$L = \{(i,j,k) \in \mathbb{Z}^3 : i + j + k \equiv 0 \bmod 2\} \tag{1.6}$$

and satisfies the following analogues of conditions (1.2)–(1.3):

(1.7) if $(i,j,k) \in \mathcal{H}$, then $(i',j',k') \in \mathcal{H}$ whenever $|i'-i| + |j'-j| \leq k'-k$;

(1.8) for any $(i',j',k') \in \mathcal{H}$, the set $\{(i,j,k) \in \mathcal{H} : |i'-i| + |j'-j| \leq k'-k\}$ is finite.

Theorem 1.6. *Let* $H_{\rm init} = \{(a,b,c) \in \mathcal{H} : (a,b,c-2) \notin \mathcal{H}\}$. *For every* $(i,j,k) \in \mathcal{H}$, *the entry* $y_{i,j,k}$ *is a Laurent polynomial with coefficients in* $\mathbb{Z}[\alpha,\beta]$ *in the initial entries* $y_{a,b,c}$, *for* $(a,b,c) \in H_{\rm init}$.

The octahedron recurrence on the half-lattice

$$\mathcal{H} = \{(i,j,k) \in L : k \geq 0\} \tag{1.9}$$

was studied by W. H. Mills, D. P. Robbins, and H. Rumsey in their pioneering work [9] on the Alternating Sign Matrix Conjecture (cf. [1] and [10, Section 10] for further references); in particular, they proved the special case of Theorem 1.6 for this choice of $\mathcal{H}$.

Example 1.7. *(Two-term version of the Gale-Robinson sequence)* Let p, q, and n be positive integers such that $p < q \leq n/2$, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_{k+n} = \frac{\alpha y_{k+p} y_{k+n-p} + \beta y_{k+q} y_{k+n-q}}{y_k}\,. \tag{1.10}$$

Using Theorem 1.6, one can prove that this sequence also exhibits the Laurent phenomenon.

Theorem 1.8. *As a function of the initial terms* $y_0, \ldots, y_{n-1}$*, every term* y_m *is a Laurent polynomial with coefficients in* $\mathbb{Z}[\alpha,\beta]$.

We note that in the special case $\alpha=\beta=1$, $p=1$, $q=2$, $n=5$ (resp., $n=4$), (1.10) becomes the Somos-5 (resp., Somos-4) recurrence [7].

The last example of the Laurent phenomenon presented in this section is of a somewhat different kind; it is inspired by [2].

Example 1.9. Let $n \geq 3$ be an integer, and consider a quadratic form

$$P(x_1,\ldots,x_n) = x_1^2+\cdots+x_n^2+\sum_{i<j}\alpha_{ij}x_ix_j \ .$$

Define the rational transformations $F_1,\ldots,F_n$ by

$$F_i:(x_1,\ldots,x_n)\mapsto \Big(x_1,\ldots,x_{i-1},\frac{P\big|_{x_i=0}}{x_i},x_{i+1},\ldots,x_n\Big). \tag{1.11}$$

Theorem 1.10. *For any sequence of indices* $i_1,\ldots,i_m$*, the composition map* $G = F_{i_1}\circ\cdots\circ F_{i_m}$ *is given by*

$$G : x=(x_1,\ldots,x_n)\mapsto (G_1(x),\ldots,G_n(x)),$$

where $G_1,\ldots,G_n$ *are Laurent polynomials with coefficients in* $\mathbb{Z}[\alpha_{ij} : i<j]$.

This paper is an outgrowth of [6], where we initiated the study of a new class of commutative algebras, called cluster algebras, and established the Laurent phenomenon in that context. Here we prove the theorems stated above, along with a number of related results, using an approach inspired by [6]. The first step is to reformulate the problem in terms of generalized *exchange patterns* (cf. [6, Definition 2.1]), which consist of *clusters* and *exchanges* among them. The clusters are distinguished finite sets of variables, each of the same cardinality n. An exchange operation on a cluster $\mathbf{x}$ replaces a variable $x\in\mathbf{x}$ by a new variable $x'=\frac{P}{x}$, where P is a polynomial in the $n-1$ variables $\mathbf{x}-\{x\}$. Each of the above theorems can be restated as saying that any member of the cluster obtained from an initial cluster $\mathbf{x}_0$ by a particular sequence of exchanges is a Laurent polynomial in the variables from $\mathbf{x}_0$. Theorem 1.10 is explicitly stated in this way; in the rest of examples above, the rephrasing is less straightforward.

Our main technical tool is "The Caterpillar Lemma" (Theorem 2.1), which establishes the Laurent phenomenon for a particular class of exchange patterns (see Figure 1). This is a modification of the namesake statement [6, Theorem 3.2], and its proof closely follows the argument in [6]. (We note that none of the two statements is a formal consequence of another.)

In most applications, including Theorems 1.2 and 1.6 above, the "caterpillar" patterns to which Theorem 2.1 applies, are not manifestly present within the original setup. Thus, we first complete it by creating additional clusters and exchanges, and then apply the Caterpillar Lemma.

The paper is organized as follows. The Caterpillar Lemma is proved in Section 2. Subsequent sections contain its applications. In particular, Theorems 1.2, 1.4, 1.6, and 1.8 are proved in Section 4, while Theorem 1.10 is proved in Section 5. Other instances of the Laurent phenomenon treated in this paper include generalizations of each of the following: Somos-4 sequences (Example 3.3), Elkies's "knight recurrence" (Example 4.1), frieze patterns (Example 4.3) and number walls (Example 4.4).

We conjecture that in all instances of the Laurent phenomenon established in this paper, the Laurent polynomials in question have *nonnegative* integer coefficients. In other contexts, similar nonnegativity conjectures were made earlier in [4, 5, 6].

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2. THE CATERPILLAR LEMMA

Let us fix an integer $n \geq 2$, and let T be a tree whose edges are labeled by the elements of the set $[n] = \{1, 2, \ldots, n\}$, so that the edges emanating from each vertex receive different labels. By a common abuse of notation, we will sometimes denote by T the set of the graph's vertices. We will write $t \overset{k}{\text{——}} t'$ if vertices $t, t' \in T$ are joined by an edge labeled by k.

From now on, let $\mathbb{A}$ be a unique factorization domain (the ring of integers $\mathbb{Z}$ or a suitable polynomial ring would suffice for most applications). Assume that a nonzero polynomial $P \in \mathbb{A}[x_1, \ldots, x_n]$, not depending on x_k, is associated with every edge $t \overset{k}{\text{——}} t'$ in T. We will write $t \underset{P}{\text{——}} t'$ or $t \overset{k}{\underset{P}{\text{——}}} t'$, and call P the *exchange polynomial* associated with the given edge. The entire collection of these polynomials is called a *generalized exchange pattern* on T. (In [6], we introduced a much narrower notion of an *exchange pattern;* hence the terminology.)

We fix a root vertex $t_0 \in T$, and introduce the *initial cluster* $\mathbf{x}(t_0)$ of n independent variables $x_1(t_0), \ldots, x_n(t_0)$. To each vertex $t \in T$, we then associate a *cluster* $\mathbf{x}(t)$ consisting of n elements $x_1(t), \ldots, x_n(t)$ of the field of rational functions $\mathbb{A}(x_1(t_0), \ldots, x_n(t_0))$. The elements $x_i(t)$ are uniquely determined by the following *exchange relations,* for every edge $t \overset{k}{\underset{P}{\text{——}}} t'$:

$$x_i(t) = x_i(t') \quad \text{for any } i \neq k; \tag{2.1}$$

$$x_k(t)\, x_k(t') = P(\mathbf{x}(t)). \tag{2.2}$$

(One can recursively compute the $x_i(t)$'s, moving away from the root. Since the exchange polynomial P does not depend on x_k, the exchange relation (2.2) does not change if we apply it in the opposite direction.)

We next introduce a special class of "caterpillar" patterns, and state conditions on their exchange polynomials that will imply Laurentness.

For $m \geq 1$, let $\mathbb{T}_{n,m}$ be the tree of the form shown in Figure 1.

The tree $\mathbb{T}_{n,m}$ has m vertices of degree n in its "spine" and $m(n-2)+2$ vertices of degree 1. We label every edge of the tree by an element of $[n]$, so that the n edges emanating from each vertex on the spine receive different labels. We let the root t_0 be a vertex in $\mathbb{T}_{n,m}$ that does not belong to the spine but is connected to one of its ends. This gives rise to the orientation of the spine, with all the arrows

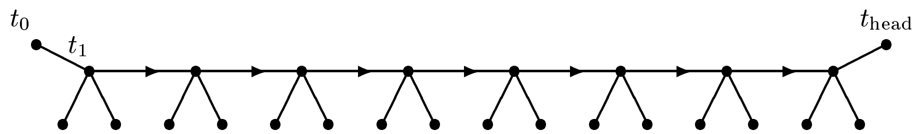


FIGURE 1. The "caterpillar" tree $\mathbb{T}_{n,m}$, for $n=4$, $m=8$

pointing away from t_0 (see Figure 1). We assign a nonzero exchange polynomial $P \in \mathbb{A}[x_1,\dots,x_n]$ to every edge t —— t' of $\mathbb{T}_{n,m}$, thus obtaining an exchange pattern.

For a rational function $F = F(x,y,\dots)$, we will denote by $F|_{x\leftarrow g(x,y,\dots)}$ the result of substituting $g(x,y,\dots)$ for x into F. To illustrate, if $F(x,y)=xy$, then $F|_{x\leftarrow \frac{y}{x}} = \frac{y^2}{x}$.

Theorem 2.1. (The Caterpillar Lemma) *Assume that a generalized exchange pattern on $\mathbb{T}_{n,m}$ satisfies the following conditions:*

(2.3) *For any edge $\bullet \overset{k}{\underset{P}{\text{——}}} \bullet$, the polynomial P does not depend on x_k, and is not divisible by any x_i, $i \in [n]$.*

(2.4) *For any two edges $\bullet \overset{i}{\underset{P}{\text{——}}} \bullet \overset{j}{\underset{Q}{\longrightarrow}} \bullet$, the polynomials P and $Q_0 = Q|_{x_i=0}$ are coprime elements of $\mathbb{A}[x_1,\dots,x_n]$.*

(2.5) *For any three edges $\bullet \overset{i}{\underset{P}{\text{——}}} \bullet \overset{j}{\underset{Q}{\longrightarrow}} \bullet \overset{i}{\underset{R}{\text{——}}} \bullet$ labeled i,j,i, we have*

$$L \cdot Q_0^b \cdot P = R\big|_{x_j \leftarrow \frac{Q_0}{x_j}},$$

where b is a nonnegative integer, $Q_0 = Q|_{x_i=0}$, and L is a Laurent monomial whose coefficient lies in $\mathbb{A}$ and is coprime with P.

Then each element $x_i(t)$, for $i \in [n]$, $t \in \mathbb{T}_{n,m}$, is a Laurent polynomial in $x_1(t_0),\dots,x_n(t_0)$, with coefficients in $\mathbb{A}$.

(Note the orientation of edges in (2.4)–(2.5).)

Proof. Our argument is essentially the same as in [6, Theorem 3.2]. For $t \in \mathbb{T}_{n,m}$, let

$$\mathcal{L}(t) = \mathbb{A}[x_1(t)^{\pm 1},\dots,x_n(t)^{\pm 1}]$$

denote the Laurent polynomial ring in the cluster $\mathbf{x}(t)$ with coefficients in $\mathbb{A}$. We view each $\mathcal{L}(t)$ as a subring of the ambient field of rational functions $\mathbb{A}(\mathbf{x}(t_0))$.

In this notation, our goal is to show that every cluster $\mathbf{x}(t)$ is contained in $\mathcal{L}(t_0)$. We abbreviate $\mathcal{L}_0 = \mathcal{L}(t_0)$. Note that $\mathcal{L}_0$ is a unique factorization domain, so any two elements $x,y \in \mathcal{L}_0$ have a well-defined greatest common divisor $\gcd(x,y)$ which is an element of $\mathcal{L}_0$ defined up to a multiple from the group $\mathcal{L}_0^\times$ of invertible elements in $\mathcal{L}_0$; the group $\mathcal{L}_0^\times$ consists of Laurent monomials in $x_1(t_0),\dots,x_n(t_0)$ whose coefficient belongs to $\mathbb{A}^\times$, the group of invertible elements of $\mathbb{A}$.

To prove that all $\mathbf{x}(t)$ are contained in $\mathcal{L}_0$, we proceed by induction on m, the size of the spine. The claim is trivial for $m=1$, so let us assume that $m \geq 2$, and furthermore assume that our statement is true for all "caterpillars" with smaller spine. It is thus enough to prove that $\mathbf{x}(t_{\text{head}}) \subset \mathcal{L}_0$, where t_{head} is one of the vertices most distant from t_0 (see Figure 1).

We assume that the path from t_0 to t_{head} starts with the following two edges: $t_0 \overset{i}{\underset{P}{\text{———}}} t_1 \xrightarrow[Q]{j} t_2$. Let $t_3 \in \mathbb{T}_{n,m}$ be the vertex such that $t_2 \overset{i}{\underset{R}{\text{———}}} t_3$. The following lemma plays a crucial role in our proof.

Lemma 2.2. *The clusters* $\mathbf{x}(t_1)$, $\mathbf{x}(t_2)$, *and* $\mathbf{x}(t_3)$ *are contained in* $\mathcal{L}_0$. *Furthermore,* $\gcd(x_i(t_3), x_i(t_1)) = \gcd(x_j(t_2), x_i(t_1)) = 1$.

Proof. The only element in the clusters $\mathbf{x}(t_1)$, $\mathbf{x}(t_2)$, and $\mathbf{x}(t_3)$ whose inclusion in $\mathcal{L}_0$ is not immediate from (2.1)–(2.2) is $x_i(t_3)$. To simplify the notation, let us denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, and $v = x_i(t_3)$, so that these variables appear in the clusters at $t_0, \ldots, t_3$, as shown below:

$$\underset{t_0}{\overset{y,x}{\bullet}} \;\overset{i}{\underset{P}{\text{————}}}\; \underset{t_1}{\overset{z,y}{\bullet}} \;\xrightarrow[Q]{\quad j \quad}\; \underset{t_2}{\overset{u,z}{\bullet}} \;\overset{i}{\underset{R}{\text{————}}}\; \underset{t_3}{\overset{v,u}{\bullet}}\ .$$

Note that the variables x_k, for $k \notin \{i, j\}$, do not change as we move among the four clusters under consideration. The lemma is then restated as saying that

$$v \in \mathcal{L}_0; \tag{2.6}$$

$$\gcd(z, u) = 1\ ; \tag{2.7}$$

$$\gcd(z, v) = 1\ . \tag{2.8}$$

Another notational convention will be based on the fact that each of the polynomials P, Q, R has a distinguished variable on which it depends, namely x_j for P and R, and x_i for Q. (In view of (2.3), P and R do not depend on x_i, while Q does not depend on x_j.) With this in mind, we will routinely write P, Q, and R as polynomials in one (distinguished) variable. For example, we rewrite the formula in (2.5) as

$$R\left(\frac{Q(0)}{y}\right) = L(y)Q(0)^b P(y), \tag{2.9}$$

where we denote $L(y) = L|_{x_j \leftarrow y}$. In the same spirit, the notation Q', R', etc., will refer to the partial derivatives with respect to the distinguished variable.

We will prove the statements (2.6), (2.7), and (2.8) one by one, in this order. We have:

$$z = \frac{P(y)}{x}\,;$$

$$u = \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y}\,;$$

$$v = \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} + \frac{R\left(\frac{Q(0)}{y}\right)}{z}\,.$$

Since

$$\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} \in \mathcal{L}_0$$

and

$$\frac{R\left(\frac{Q(0)}{y}\right)}{z} = \frac{L(y)Q(0)^b P(y)}{z} = L(y)Q(0)^b x \in \mathcal{L}_0\,,$$

(2.6) follows.

We next prove (2.7). We have

$$u = \frac{Q(z)}{y} \equiv \frac{Q(0)}{y} \bmod z\,.$$

Since x and y are invertible in $\mathcal{L}_0$, we conclude that $\gcd(z,u) = \gcd(P(y), Q(0)) = 1$ (using (2.4)).

It remains to prove (2.8). Let

$$f(z) = R\left(\tfrac{Q(z)}{y}\right).$$

Then

$$v = \frac{f(z) - f(0)}{z} + L(y)Q(0)^b x\,.$$

Working mod$z$, we obtain:

$$\frac{f(z) - f(0)}{z} \equiv f'(0) = R'\left(\tfrac{Q(0)}{y}\right) \cdot \tfrac{Q'(0)}{y}\,.$$

Hence

$$v \equiv R'\left(\tfrac{Q(0)}{y}\right) \cdot \tfrac{Q'(0)}{y} + L(y)Q(0)^b x \bmod z\,.$$

Note that the right-hand side is a polynomial of degree 1 in x whose coefficients are Laurent polynomials in the rest of the variables of the cluster $\mathbf{x}(t_0)$. Thus (2.8) follows from $\gcd\big(L(y)Q(0)^b, P(y)\big) = 1$, which is a consequence of (2.4)–(2.5). □

We can now complete the proof of Theorem 2.1. We need to show that any variable $X = x_k(t_{\rm head})$ belongs to $\mathcal{L}_0$. Since both t_1 and t_3 are closer to $t_{\rm head}$ than t_0, we can use the inductive assumption to conclude that X belongs to both $\mathcal{L}(t_1)$ and $\mathcal{L}(t_3)$. Since $X \in \mathcal{L}(t_1)$, it follows from (2.1) that X can be written as $X = f/x_i(t_1)^a$ for some $f \in \mathcal{L}_0$ and $a \in \mathbb{Z}_{\geq 0}$. On the other hand, since $X \in \mathcal{L}(t_3)$, it follows from (2.1) and from the inclusion $x_i(t_3) \in \mathcal{L}_0$ provided by Lemma 2.2 that X has the form $X = g/x_j(t_2)^b x_i(t_3)^c$ for some $g \in \mathcal{L}_0$ and some $b, c \in \mathbb{Z}_{\geq 0}$. The inclusion $X \in \mathcal{L}_0$ now follows from the fact that, by the last statement in Lemma 2.2, the denominators in the two obtained expressions for X are coprime in $\mathcal{L}_0$. □

3. One-dimensional recurrences

In this section, we apply Theorem 2.1 to study the Laurent phenomenon for sequences $y_0, y_1, \ldots$ given by recursions of the form

$$y_{m+n} y_m = F(y_{m+1}, \ldots, y_{m+n-1}), \tag{3.1}$$

where $F \in \mathbb{A}[x_1, \ldots, x_{n-1}]$.

For an integer m, let $\langle m \rangle$ denote the unique element of $[n] = \{1, \ldots, n\}$ satisfying $m \equiv \langle m \rangle \bmod n$. We define the polynomials $F_1, \ldots, F_n \in \mathbb{A}[x_1, \ldots, x_n]$ by

$$F_m = F(x_{\langle m+1 \rangle}, x_{\langle m+2 \rangle}, \ldots, x_{\langle m-1 \rangle}); \tag{3.2}$$

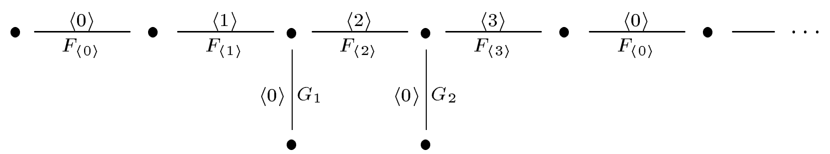


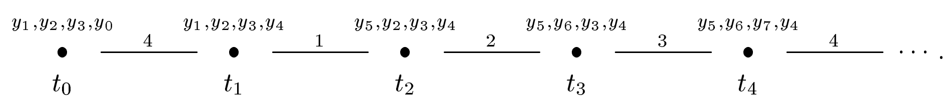
FIGURE 2. Constructing a caterpillar; $n = 4$.

thus F_m does not depend on the variable x_m. We introduce the infinite "cyclic exchange pattern"

$$t_0 \ \frac{\langle 0\rangle}{F_{\langle 0\rangle}} \ t_1 \ \frac{\langle 1\rangle}{F_{\langle 1\rangle}} \ t_2 \ \frac{\langle 2\rangle}{F_{\langle 2\rangle}} \ t_3 \ \frac{\langle 3\rangle}{F_{\langle 3\rangle}} \ t_4 \ \text{——} \ \cdots, \tag{3.3}$$

and let the cluster at each point t_m consist of the variables $y_m, \dots, y_{m+n-1}$, labeled within the cluster according to the rule $y_s = x_{\langle s\rangle}(t_m)$. Then equations (3.1) become the exchange relations associated with this pattern.

To illustrate, let $n = 4$. Then the clusters will look like this:



In order to include this situation into the setup of Section 2 (cf. Figure 1), we create an infinite "caterpillar tree" whose "spine" is formed by the vertices t_m, $m > 0$. We thus attach the missing $n-2$ "legs" with labels in $[n] - \{\langle m-1\rangle, \langle m\rangle\}$, to each vertex t_m.

Our next goal is to state conditions on the polynomial F which make it possible to assign exchange polynomials satisfying (2.3)–(2.5) to the newly constructed legs. The first requirement (cf. (2.3)) is:

(3.4) The polynomial F is not divisible by any x_i, $i \in [n-1]$.

For $m \in [n-1]$, we set

$$Q_m = F_m|_{x_n \leftarrow 0} = F(x_{m+1}, \dots, x_{n-1}, 0, x_1, \dots, x_{m-1}). \tag{3.5}$$

Our second requirement is

(3.6) Each Q_m is an irreducible element of $\mathbb{A}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$.

To state our most substantial requirement, we recursively define a sequence of polynomials $G_{n-1}, \dots, G_1, G_0 \in \mathbb{A}[x_1, \dots, x_{n-1}]$; more precisely, each G_m will be defined up to a multiple in $\mathbb{A}^\times$. (Later, $G_1, \dots, G_{n-2}$ will become the exchange polynomials assigned to the "legs" of the caterpillar labeled by $n = \langle 0\rangle$; see Figure 2.)

We set $G_{n-1} = F$, and obtain each G_{m-1} from G_m, as follows. Let

$$\widetilde{G}_{m-1} = G_m\Big|_{x_m \leftarrow \frac{Q_m}{x_m}}. \tag{3.7}$$

Let L be a Laurent monomial in $x_1, \dots, x_{n-1}$, with coefficient in $\mathbb{A}$, such that

$$\widetilde{\widetilde{G}}_{m-1} = \frac{\widetilde{G}_{m-1}}{L} \tag{3.8}$$

is a polynomial in $\mathbb{A}[x_1, \dots, x_{n-1}]$ not divisible by any x_i or by any non-invertible scalar in $\mathbb{A}$. Such an L is unique up to a multiple in $A^\times$. Finally, we set

$$G_{m-1} = \frac{\widetilde{\widetilde{G}}_{m-1}}{Q_m^b}, \tag{3.9}$$

where Q_m^b is the maximal power of Q_m that divides $\widetilde{\widetilde{G}}_{m-1}$. With all this notation, our final requirement is:

$$G_0 = F. \tag{3.10}$$

Theorem 3.1. *Let F be a polynomial in the variables $x_1, \dots, x_{n-1}$ with coefficients in a unique factorization domain $\mathbb{A}$ satisfying conditions* (3.4), (3.6), and (3.10). *Then every term of the sequence (y_i) defined by the recurrence*

$$y_{m+n} = \frac{F(y_{m+1}, \dots, y_{m+n-1})}{y_m}$$

is a Laurent polynomial in the initial n terms, with coefficients in $\mathbb{A}$.

Proof. To prove the Laurentness of some y_N, we will apply Theorem 2.1 to the caterpillar tree constructed as follows. We set $t_{\text{head}} = t_{N-n+1}$; this corresponds to the first cluster containing y_N. As a path from t_0 to t_{head}, we take a finite segment of (3.3):

$$t_0 \;\frac{\langle 0\rangle}{F_{\langle 0\rangle}}\; t_1 \;\frac{\langle 1\rangle}{F_{\langle 1\rangle}}\; t_2 \;\frac{\langle 2\rangle}{F_{\langle 2\rangle}}\; \cdots \;\frac{\langle N-1\rangle}{F_{\langle N-1\rangle}}\; t_{N-n} \;\frac{\langle N\rangle}{F_{\langle N\rangle}}\; t_{N-n+1}\,. \tag{3.11}$$

We then define the exchange polynomial $G_{j,k-1}$ associated with the leg labeled j attached to a vertex t_k on the spine (see Figure 3) by

$$G_{j,k-1} = G_{\langle k-j-1\rangle}(x_{\langle j+1\rangle}, \dots, x_n, x_1, \dots, x_{\langle j-1\rangle}),$$

where in the right-hand side, we use the polynomials $G_1, \dots, G_{n-2}$ constructed in (3.7)–(3.9) above.

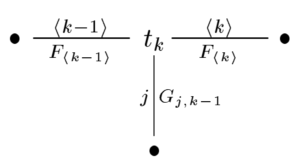


FIGURE 3

It remains to verify that this exchange pattern satisfies (2.3), (2.4), and (2.5). Condition (2.3) for the edges appearing in (3.11) is immediate from (3.4), while for the rest of the edges, it follows from the definition of $\widetilde{\widetilde{G}}_{m-1}$ in (3.8).

Turning to (2.4), we first note that we may assume $i = \langle 0\rangle = n$ (otherwise apply a cyclic shift of indices). Under this assumption, we can identify the polynomials P and Q_0 in (2.4) with the polynomials G_{m-1} and Q_m in (3.9), for some value of m. (The special case of P attached to one of the edges in (3.11) corresponds to $m = 1$, and its validity requires (3.10).) Then the condition $\gcd(G_{m-1}, Q_m) = 1$ follows from (3.6) and the choice of the exponent b in (3.9).

Finally, (2.5) is ensured by the construction (3.7)–(3.9), which was designed expressly for this purpose. As before, the special case of P attached to one of the edges in (3.11) holds due to (3.10). □

In the rest of this section, we give a few applications of Theorem 3.1. In all of them, conditions (3.4) and (3.6) are immediate, so we concentrate on the verification of (3.10).

Example 3.2. Let a and b be positive integers, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_k = \frac{y_{k-2}^a y_{k-1}^b + 1}{y_{k-3}}\,.$$

We claim that every term of the sequence is a Laurent polynomial over $\mathbb{Z}$ in y_0, y_1, and y_2. To prove this, we set $n = 3$ and construct the polynomials G_2, G_1, and G_0 using (3.7)–(3.9). Initializing $G_2 = F(x_1, x_2) = x_1^a x_2^b + 1$, we obtain:

$$\begin{aligned}
&Q_2 = F(0, x_1) = 1, \qquad \tilde{G}_1 = F\big|_{x_2 \leftarrow \frac{Q_2}{x_2}} = x_1^a x_2^{-b} + 1, \qquad G_1 = \tilde{\tilde{G}}_1 = x_1^a + x_2^b,\\
&Q_1 = F(x_2, 0) = 1, \qquad \tilde{G}_0 = G_1\big|_{x_1 \leftarrow \frac{Q_1}{x_1}} = x_1^{-a} + x_2^b, \qquad G_0 = \tilde{\tilde{G}}_0 = 1 + x_1^a x_2^b = F,
\end{aligned}$$

as desired.

Example 3.3. *(Generalized Somos-4 sequence)* Let a, b, and c be positive integers, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_k = \frac{y_{k-3}^a y_{k-1}^c + y_{k-2}^b}{y_{k-4}}\,.$$

(The Somos-4 sequence [7], introduced by Michael Somos, is the special case $a = c = 1$, $b = 2$.) Again, each y_i is a Laurent polynomial in the initial terms y_0, y_1, y_2, and y_3. To prove this, we set $n = 4$ and compute $G_3, \ldots, G_0$ using (3.7)–(3.9) and beginning with $G_3 = F = x_1^a x_3^c + x_2^b$:

$$\begin{aligned}
&Q_3 = F(0, x_1, x_2) = x_1^b, \qquad G_3\big|_{x_3 \leftarrow \frac{Q_3}{x_3}} = x_1^{a+bc} x_3^{-c} + x_2^b, \qquad G_2 = x_1^{a+bc} + x_2^b x_3^c,\\
&Q_2 = F(x_3, 0, x_1) = x_1^c x_3^a, \qquad G_2\big|_{x_2 \leftarrow \frac{Q_2}{x_2}} = x_1^{a+bc} + x_1^{bc} x_2^{-b} x_3^{ab+c}, \qquad G_1 = x_1^a x_2^b + x_3^{ab+c},\\
&Q_1 = F(x_2, x_3, 0) = x_3^b, \qquad G_1\big|_{x_1 \leftarrow \frac{Q_1}{x_1}} = x_1^{-a} x_2^b x_3^{ab} + x_3^{ab+c}, \qquad G_0 = x_2^b + x_1^a x_3^c = F,
\end{aligned}$$

and the claim follows.

Remark 3.4. The Laurent phenomena in Theorems 1.4 and 1.8 can also be proved by applying Theorem 3.1: in the former (resp., latter) case, the polynomial F is given by $F = \alpha x_p x_{n-p} + \beta x_q x_{n-q} + \gamma x_r x_{n-r}$ (resp., $F = \alpha x_p x_{n-p} + \beta x_q x_{n-q}$). The proofs are straightforward but rather long. Shorter proofs, based on J. Propp's idea of viewing one-dimensional recurrences as "projections" of multi-dimensional ones, are given in Section 4 below.

4. Two- and three-dimensional recurrences

In this section, we use the strategy of Section 3 to establish the Laurent phenomenon for several recurrences involving two- and three-dimensional arrays. Our first example generalizes a construction (and the corresponding Laurentness conjecture) suggested by Noam Elkies and communicated by James Propp. Even though the Laurent phenomenon in this example can be deduced from Theorem 1.6, we choose to give a self-contained treatment, for the sake of exposition.

Example 4.1. *(The knight recurrence)* Consider a two-dimensional array $(y_{ij})_{i,j\geq 0}$ whose entries satisfy the recurrence

$$y_{i,j}y_{i-2,j-1} = \alpha y_{i,j-1}y_{i-2,j} + \beta y_{i-1,j}y_{i-1,j-1}\,. \tag{4.1}$$

We will prove that every y_{ij} is a Laurent polynomial in the initial entries

$$Y_{\rm init} = \{y_{ab} \,:\, a < 2 \text{ or } b < 1\},$$

with coefficients in the ring $\mathbb{A} = \mathbb{Z}[\alpha, \beta]$.

We will refer to $Y_{\rm init}$ as the *initial cluster,* even though it is an infinite set. Notice, however, that each individual y_{ij} only depends on finitely many variables $\{y_{ab} \in Y_{\rm init} \,:\, a \leq i,\ b \leq j\}$.

Similarly to Section 3, we will use the exchange relations (4.1) to create a sequence of clusters satisfying the Caterpillar Lemma (Theorem 2.1).

This is done in the following way. Let us denote by $\mathcal{H} = \mathbb{Z}^2_{\geq 0}$ the underlying set of indices; for $h = (i,j) \in \mathcal{H}$, we will write $y_h = y_{ij}$. The variables of the initial cluster have labels in the set

$$H_{\rm init} = \{(i,j) \in \mathcal{H} \,:\, i < 2 \text{ or } j < 1\}.$$

In Figure 4, the elements of $H_{\rm init}$ are marked by $\bullet$'s.

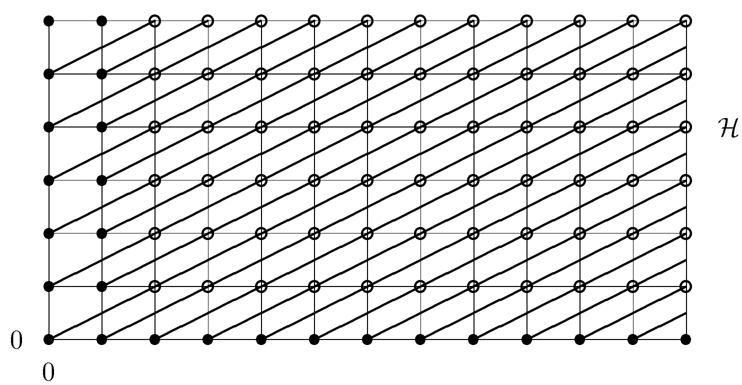


Figure 4. The initial cluster and the equivalence classes $\langle h\rangle$

We introduce the product partial order on $\mathcal{H}$:

$$(i_1, j_1) \leq (i_2, j_2) \overset{\rm def}{\Leftrightarrow} (i_1 \leq i_2) \text{ and } (j_1 \leq j_2). \tag{4.2}$$

For an element $h = (i,j) \in \mathcal{H} - H_{\rm init}$, let us denote $h^- = (i-2, j-1)$; in this notation, the exchange relation (4.1) expresses the product $y_h \cdot y_{h^-}$ as a polynomial in the variables $y_{h'}$, for $h^- < h' < h$.

We write $h^- \sim h$, and extend this to an equivalence relation $\sim$ on $\mathcal{H}$. The equivalence class of h is denoted by $\langle h \rangle$. These classes are shown as slanted lines in Figure 4. All our exchange polynomials will belong to the ring $\mathbb{A}[x_a : a \in \mathcal{H}/\!\sim]$.

Note that $H_{\rm init}$ has exactly one representative from each equivalence class. We will now construct a sequence of subsets $H_0 = H_{\rm init}, H_1, H_2, \dots$, each having this property, using the following recursive rule. Let us fix a particular linear extension of the partial order (4.2), say,

$$(i_1, j_1) \preceq (i_2, j_2) \stackrel{\rm def}{\Leftrightarrow} (i_1 + j_1 < i_2 + j_2) \text{ or } (i_1 + j_1 = i_2 + j_2 \text{ and } i_1 \le i_2).$$

Restricting this linear ordering to the complement $\mathcal{H} - H_{\rm init}$ of the initial cluster, we obtain a numbering of the elements of this complement by positive integers:

$$\begin{aligned} &h_0 = (2,1),\ h_1 = (2,2),\ h_2 = (3,1),\ h_3 = (2,3),\ h_4 = (3,2),\\ &h_5 = (4,1),\ h_6 = (2,4),\ h_7 = (3,3),\ h_8 = (4,2), \end{aligned}$$

and so on. Having constructed H_m, we let $H_{m+1} = H_m \cup \{h_m\} - \{h_m^-\}$. To illustrate, the set H_9 is shown in Figure 5.

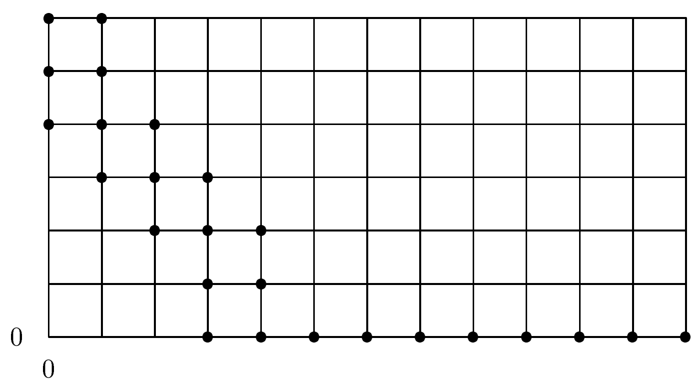


FIGURE 5. Indexing set H_9

We next create the infinite exchange pattern

$$(4.3) \qquad t_0 \xrightarrow[P_{\langle h_0 \rangle}]{\langle h_0 \rangle} t_1 \xrightarrow[P_{\langle h_1 \rangle}]{\langle h_1 \rangle} t_2 \xrightarrow[P_{\langle h_2 \rangle}]{\langle h_2 \rangle} t_3 \xrightarrow[P_{\langle h_3 \rangle}]{\langle h_3 \rangle} t_4 \longrightarrow \cdots .$$

(cf. (3.3)) The cluster at each point t_m is given by $\mathbf{x}(t_m) = \{y_h : h \in H_m\}$; as before, each cluster variable y_h corresponds to the variable $x_{\langle h \rangle}$. The exchange polynomial $P_{\langle h \rangle}$ for an edge $\bullet \overset{\langle h \rangle}{\text{—}} \bullet$ with $h = (i, j)$ is given by

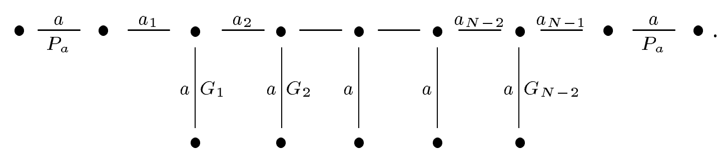
$$(4.4) \qquad P_{\langle h \rangle} = \alpha x_{\langle (i,j-1) \rangle} x_{\langle (i-2,j) \rangle} + \beta x_{\langle (i-1,j) \rangle} x_{\langle (i-1,j-1) \rangle}.$$

Then equations (4.1) become the exchange relations associated with this pattern.

To establish the Laurent phenomenon, we will complete the caterpillar pattern by attaching "legs" to each vertex t_m and assigning exchange polynomials to these legs so that the appropriate analogues of conditions (3.4), (3.6) and (3.10) are satisfied. Since we now work over the polynomial ring $\mathbb{A}[x_a : a \in \mathcal{H}/\!\sim]$ in infinitely many indeterminates, the number of legs attached to every vertex t_m will also be infinite (one for every label a different from $\langle h_{m-1} \rangle$ and $\langle h_m \rangle$). This will not matter much for our argument though: to prove the Laurentness for any y_{h_m}, we will simply restrict our attention to the finite part of the infinite caterpillar tree lying between t_0 and $t_{\text{head}} = t_{m+1}$, and to the legs labeled by $\langle h_k\rangle$ for $0 \le k \le m$.

The role of conditions (3.4) and (3.6) is now played by the observation that each exchange polynomial $P_{\langle h\rangle}$ is not divisible by any variable x_a, and furthermore every specialization $P_{\langle h\rangle}\big|_{x_a\leftarrow 0}$ is an irreducible element of the Laurent polynomial ring.

To formulate the analogue of (3.10), let us fix an equivalence class $a \in \mathcal{H}/\!\sim$ and concentrate on defining the exchange polynomials for the legs labeled by a and attached to the vertices squeezed between two consecutive occurrences of the label a on the spine:

(4.5) 

We note that the labels $a_1, \dots, a_{N-1} \in \mathcal{H}/\!\sim$ appearing on the spine between these two occurrences of a are distinct. For $m = N-2, N-3, \dots, 1$, we denote by G_m the exchange polynomial to be associated with the a-labeled leg attached between the edges labeled a_m and a_{m+1} (cf. (4.5)).

The polynomials G_m are defined with the help of a recursive procedure analogous to (3.7)–(3.9). We initialize $G_{N-1} = P_a$, and obtain each G_{m-1} from G_m, as follows. The step (3.7) is replaced by

$$\tilde{G}_{m-1} = G_m\Big|_{x_{a_m} \leftarrow \frac{Q_m}{x_{a_m}}}$$

with

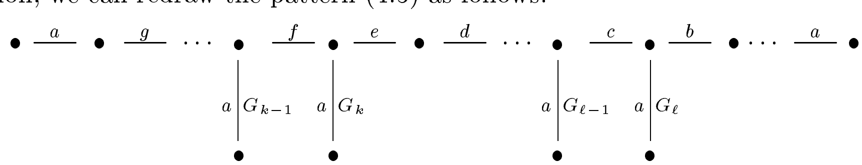
$$Q_m = P_{a_m}\Big|_{x_a\leftarrow 0}\,. \tag{4.6}$$

We then compute $\tilde{\tilde{G}}_{m-1}$ and G_{m-1} exactly as in (3.8)–(3.9). By the argument given in the proof of Theorem 3.1, the equality $G_0 = P_a$ would imply the desired Laurentness (cf. (3.10)).

To simplify computations, we denote the equivalence classes "surrounding" a, as shown below:

$$\begin{array}{ccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & q & p & f & c & a & \cdots \\ \cdots & f & c & a & e & b & \cdots \\ \cdots & a & e & b & g & d & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\,. \tag{4.7}$$

In other words, if $a = \langle(i,j)\rangle$, then $b = \langle(i,j-1)\rangle$, $c = \langle(i-1,j)\rangle$, etc. With this notation, we can redraw the pattern (4.5) as follows:

(4.8) 

for appropriate values of k and ℓ.

We will call a value of m *essential* if $G_{m-1} \neq G_m$. We are going to see that the essential values of m are those for which $a_m \in \{b, c, e, f\}$; in the notation of (4.8), these values are $\ell+1$, ℓ, $k+1$, and k.

We initialize $G_{N-1} = P_a = \alpha x_b x_f + \beta x_c x_e$. The values of m in the interval $\ell < m < N$ are not essential since the variable x_{a_m} does not enter P_a, which is furthermore not divisible by Q_m (because the latter involves variables absent in P_a).

The first essential value is $m = \ell+1$, with $a_m = b$:

$$Q_{\ell+1} = P_b|_{x_a \leftarrow 0} = (\alpha x_a x_d + \beta x_e x_g)|_{x_a \leftarrow 0} = \beta x_e x_g ,$$

$$\widetilde{G}_\ell = P_a\Big|_{x_b \leftarrow \frac{Q_{\ell+1}}{x_b}} = \alpha \frac{\beta x_e x_g}{x_b} x_f + \beta x_c x_e ,$$

$$G_\ell = \alpha x_g x_f + x_b x_c .$$

Step $m = \ell$ (here $a_m = c$):

$$Q_\ell = P_c|_{x_a \leftarrow 0} = (\alpha x_e x_p + \beta x_a x_f)|_{x_a \leftarrow 0} = \alpha x_e x_p ,$$

$$\widetilde{G}_{\ell-1} = G_\ell\Big|_{x_c \leftarrow \frac{Q_\ell}{x_c}} = \alpha x_g x_f + x_b \frac{\alpha x_e x_p}{x_c} ,$$

$$G_{\ell-1} = x_c x_g x_f + x_b x_e x_p .$$

Notice that $G_{\ell-1}$ does not involve x_d, so the value $m = k+2$ is not essential, as are the rest of the values in the interval $k+1 < m < \ell$.

Step $m = k+1$, with $a_m = e$:

$$Q_{k+1} = P_e|_{x_a \leftarrow 0} = (\alpha x_c x_g + \beta x_a x_b)|_{x_a \leftarrow 0} = \alpha x_c x_g ,$$

$$\widetilde{G}_k = x_c x_g x_f + x_b x_p \frac{\alpha x_c x_g}{x_e} ,$$

$$G_k = x_f x_e + \alpha x_b x_p .$$

Step $m = k$, with $a_m = f$:

$$Q_k = P_f|_{x_a \leftarrow 0} = (\alpha x_a x_q + \beta x_c x_p)|_{x_a \leftarrow 0} = \beta x_c x_p ,$$

$$\widetilde{G}_{k-1} = \frac{\beta x_c x_p}{x_f} x_e + \alpha x_b x_p ,$$

$$G_{k-1} = \beta x_c x_e + \alpha x_b x_f .$$

The values of m in the interval $0 < m < k$ are not essential since none of the corresponding variables x_{a_m} appears in G_{k-1}; in particular, $m = 1$ is not essential, since G_{k-1} does not involve x_g. Hence

$$G_0 = G_{k-1} = \beta x_c x_e + \alpha x_b x_f = P_a ,$$

as desired. The Laurentness is proved.

Remark 4.2. The Laurent phenomenon for the recurrence (4.1) actually holds in greater generality. Specifically, one can replace $\mathcal{H}$ by any subset of $\mathbb{Z}^2$ which satisfies the following analogues of conditions (1.2)–(1.3) and (1.7)–(1.8):

(4.9) if $h \in \mathcal{H}$, then $h' \in \mathcal{H}$ whenever $h \leq h'$;

(4.10) for any $h' \in \mathcal{H}$, the set $\{h \in \mathcal{H} : h \leq h'\}$ is finite.

Then take $H_{\text{init}} = \{h \in \mathcal{H} : h^- \notin \mathcal{H}\}$.

The proof of Laurentness only needs one adjustment, concerning the choice of a linear extension $\prec$. Specifically, while proving that y_h is given by a Laurent polynomial, take a finite set $\mathcal{H}^{(h)} \subset \mathcal{H}$ containing h and satisfying the conditions

(4.11) if $h' \in \mathcal{H}^{(h)}$, then $h'' \in \mathcal{H}^{(h)}$ whenever $h'' \leq h'$ and $h'' \in \mathcal{H}$;

(4.12) for any $h' \in \mathcal{H}$ such that $h' \leq h$, there exists $h'' \in \mathcal{H}^{(h)}$ such that $h'' \geq h$ and $h'' \sim h$.

(The existence of $\mathcal{H}^{(h)}$ follows from (4.9)–(4.10).) Then define $\preceq$ exactly as before on the set $\mathcal{H}^{(h)}$; set $h' \prec h''$ for any $h' \in \mathcal{H}^{(h)}$ and $h'' \in \mathcal{H} - \mathcal{H}^{(h)}$; and define $\preceq$ on the complement $\mathcal{H} - \mathcal{H}^{(h)}$ by an arbitrary linear extension of $\leq$. These conditions ensure that the sets H_m needed in the proof of Laurentness of the given y_h are well defined, and that the rest of the proof proceeds smoothly.

Armed with the techniques developed above in this section, we will now prove the main theorems stated in the introduction.

Proof of Theorem 1.2. Our argument is parallel to that in Example 4.1, so we skip the steps which are identical in both proofs. For simplicity of exposition, we present the proof in the special case $\mathcal{H} = \mathbb{Z}^3_{\geq 0}$; the case of general $\mathcal{H}$ requires the same adjustments as those described in Remark 4.2.

We define the product partial order $\leq$ and a compatible linear order $\preceq$ on $\mathcal{H}$ by

$$\begin{aligned}
(i_1, j_1, k_1) \leq (i_2, j_2, k_2) &\overset{\text{def}}{\Leftrightarrow} (i_1 \leq i_2) \text{ and } (j_1 \leq j_2) \text{ and } (k_1 \leq k_2),\\
(i_1, j_1, k_1) \preceq (i_2, j_2, k_2) &\overset{\text{def}}{\Leftrightarrow} (i_1 + j_1 + k_1 < i_2 + j_2 + k_2)\\
&\quad \text{or } (i_1 + j_1 + k_1 = i_2 + j_2 + k_2 \text{ and } i_1 + j_1 < i_2 + j_2)\\
&\quad \text{or } (i_1 + j_1 = i_2 + j_2 \text{ and } k_1 = k_2 \text{ and } i_1 \leq i_2).
\end{aligned}$$

For $h = (i, j, k)$, we set $h^- = (i-1, j-1, k-1)$; thus, the exchange relation (1.1) expresses the product $y_h \cdot y_{h^-}$ as a polynomial in the variables $y_{h'}$, for $h^- < h' < h$.

All the steps in Example 4.1 leading to the creation of the infinite exchange pattern (4.3) are repeated verbatim. Instead of (4.4), the exchange polynomials $P_{\langle h \rangle}$ along the spine are now given by

$$\begin{aligned}
&P_{\langle (i,j,k) \rangle}\\
&= \alpha x_{\langle (i-1,j,k) \rangle} x_{\langle (i,j-1,k-1) \rangle} + \beta x_{\langle (i,j-1,k) \rangle} x_{\langle (i-1,j,k-1) \rangle} + \gamma x_{\langle (i,j,k-1) \rangle} x_{\langle (i-1,j-1,k) \rangle}.
\end{aligned}$$

The role of (4.7) is now played by Figure 6, which shows the "vicinity" of an equivalence class a. This figure displays the orthogonal projection of $\mathcal{H}$ along the vector $(1, 1, 1)$. Thus the vertices represent equivalence classes in $\mathcal{H}/\sim$. For example, if $a = \langle (i, j, k) \rangle$, then

$$\begin{aligned}
&b = \langle (i, j, k-1) \rangle, && c = \langle (i, j-1, k) \rangle, && d = \langle (i-1, j, k) \rangle,\\
&e = \langle (i, j-1, k-1) \rangle, && f = \langle (i-1, j, k-1) \rangle, && g = \langle (i-1, j-1, k) \rangle.
\end{aligned}$$

With this notation, we have:

$$P_a = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g .$$

With the polynomials $G_1, G_2, \ldots$ defined as in (4.5), the essential values of m are now those for which $a_m \in \{b, c, d, e, f, g\}$. (The verification that the rest of the values are not essential is left to the reader.) We denote these values by $m_1, \ldots, m_6$, respectively.

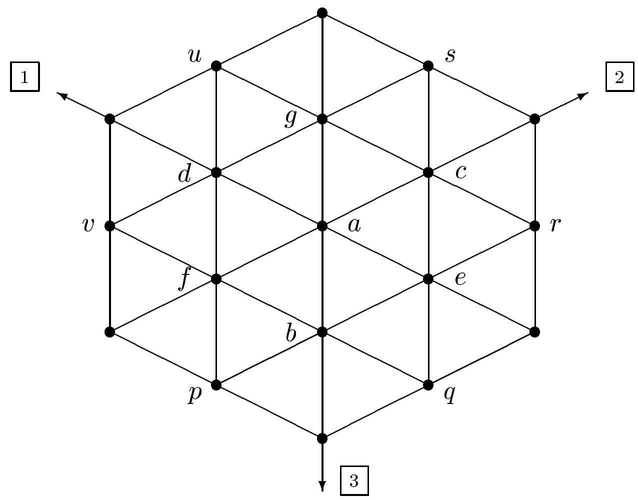


FIGURE 6. The cube recurrence

The computation of the polynomials G_m begins by initializing

$$G_{N-1} = P_a = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g \,.$$

Step $m = m_1$, $a_m = b$:

$$\begin{aligned} Q_{m_1} =&\ P_b|_{x_a \leftarrow 0} = \alpha x_f x_q + \beta x_e x_p \,; \\ \widetilde{G}_{m_1-1} =&\ G_{m_1}\Big|_{x_b \leftarrow \frac{Q_{m_1}}{b}} = \alpha x_d x_e + \beta x_c x_f + \gamma \tfrac{\alpha x_f x_q + \beta x_e x_p}{x_b} x_g \,; \\ G_{m_1-1} =&\ \alpha x_b x_d x_e + \beta x_b x_c x_f + \alpha\gamma x_f x_g x_q + \beta\gamma x_e x_g x_p \,. \end{aligned}$$

Step $m = m_2$, $a_m = c$:

$$\begin{aligned} Q_{m_2} =&\ P_c|_{x_a \leftarrow 0} = \alpha x_g x_r + \gamma x_e x_s ; \\ \widetilde{G}_{m_2-1} =&\ \alpha x_b x_d x_e + \beta x_b \tfrac{\alpha x_g x_r + \gamma x_e x_s}{x_c} x_f + \alpha\gamma x_f x_g x_q + \beta\gamma x_e x_g x_p \,; \\ G_{m_2-1} =&\ \alpha x_b x_c x_d x_e + \alpha\beta x_b x_f x_g x_r + \beta\gamma x_b x_e x_f x_s + \alpha\gamma x_c x_f x_g x_q + \beta\gamma x_c x_e x_g x_p \,. \end{aligned}$$

Step $m = m_3$, $a_m = d$:

$$\begin{aligned} Q_{m_3} =&\ P_d|_{x_a \leftarrow 0} = \beta x_g x_v + \gamma x_f x_u \,; \\ \widetilde{G}_{m_3-1} =&\ \alpha x_b x_c \tfrac{\beta x_g x_v + \gamma x_f x_u}{x_d} x_e \\ &+ \alpha\beta x_b x_f x_g x_r + \beta\gamma x_b x_e x_f x_s + \alpha\gamma x_c x_f x_g x_q + \beta\gamma x_c x_e x_g x_p \,; \\ G_{m_3-1} =&\ \alpha\beta x_b x_c x_e x_g x_v + \alpha\gamma x_b x_c x_e x_f x_u + \beta\gamma x_b x_d x_e x_f x_s + \beta\gamma x_c x_d x_e x_g x_p \\ &+ \alpha\beta x_b x_d x_f x_g x_r + \alpha\gamma x_c x_d x_f x_g x_q \,. \end{aligned}$$

Step $m = m_4$, $a_m = e$:

$$
\begin{aligned}
Q_{m_4} =&\ P_e|_{x_a \leftarrow 0} = \beta x_b x_r + \gamma x_c x_q\,;\\
\widetilde{G}_{m_4-1} =&\ \tfrac{Q_{m_4}}{x_e}(\alpha\beta x_b x_c x_g x_v + \alpha\gamma x_b x_c x_f x_u + \beta\gamma x_b x_d x_f x_s + \beta\gamma x_c x_d x_g x_p)\\
&+ \alpha x_d x_f x_g Q_{m_4}\,;\\
G_{m_4-1} =&\ \alpha\gamma x_b x_c x_f x_u + \beta\gamma x_b x_d x_f x_s + \alpha x_d x_e x_f x_g\\
&+ \alpha\beta x_b x_c x_g x_v + \beta\gamma x_c x_d x_g x_p\,.
\end{aligned}
$$

Step $m = m_5$, $a_m = f$:

$$
\begin{aligned}
Q_{m_5} =&\ P_f|_{x_a \leftarrow 0} = \alpha x_b x_v + \gamma x_d x_p\,;\\
\widetilde{G}_{m_5-1} =&\ \tfrac{Q_{m_5}}{x_f}(\alpha\gamma x_b x_c x_u + \beta\gamma x_b x_d x_s + \alpha x_d x_e x_g) + \beta x_c x_g Q_{m_5}\,;\\
G_{m_5-1} =&\ \alpha x_d x_e x_g + \beta x_c x_f x_g + \alpha\gamma x_b x_c x_u + \beta\gamma x_b x_d x_s\,.
\end{aligned}
$$

Step $m = m_6$, $a_m = g$:

$$
\begin{aligned}
Q_{m_6} =&\ P_g|_{x_a \leftarrow 0} = \alpha x_c x_u + \beta x_d x_s\,;\\
\widetilde{G}_{m_6-1} =&\ \tfrac{Q_{m_6}}{x_g}(\alpha x_d x_e + \beta x_c x_f) + \gamma x_b Q_{m_6}\,;\\
G_{m_6-1} =&\ \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g = P_a\,,
\end{aligned}
$$

completing the proof. □

We will now deduce the Gale-Robinson conjecture from Theorem 1.2.

Proof of Theorem 1.4. To prove the Laurentness of a given element y_N of the Gale-Robinson sequence (y_m), we define the array $(z_{ijk})_{(i,j,k)\in\mathcal{H}}$ by setting $z_{ijk} = y_{N+pi+qj+rk}$, with the indexing set

$$\mathcal{H} = \mathcal{H}(N) = \{(i,j,k) \in \mathbb{Z}^3 \,:\, N + pi + qj + rk \geq 0\}\,.$$

Then (1.4) implies that the z_{ijk} satisfy the cube recurrence (1.1). Note that $\mathcal{H}$ satisfies the conditions (1.2)–(1.3). Thus Theorem 1.2 applies to (z_{ijk}), with $H_{\text{init}} = \{(a,b,c) \in \mathbb{Z}^3 : 0 \leq N + pa + qb + rc < n\}$. It remains to note that $y_N = z_{000}$, while for any $(a,b,c) \in H_{\text{init}}$, we have $z_{abc} = y_m$ with $0 \leq m < n$. □

Proof of Theorem 1.6. This theorem is proved by the same argument as Theorem 1.2. We treat the Mills-Robbins-Rumsey special case (1.9) (cf. also (1.6)); similarly to Theorem 1.2, the case of general $\mathcal{H}$ requires the standard adjustments described in Remark 4.2. We use the partial order on the lattice L defined by

$$(i,j,k) \leq (i',j',k') \,:\, |i'-i| + |j'-j| \leq k'-k\,.$$

For $h = (i,j,k) \in L$, we set $h^- = (i,j,k-2)$, and define the equivalence relation $\sim$ accordingly. Figure 7 shows equivalence classes "surrounding" a given class a (cf. Figure 6).

The initialization polynomial $G_{N-1} = P_a$ is given by $P_a = \alpha x_c x_d + \beta x_b x_e$. The table below displays a_m, Q_m, $\widetilde{G}_{m-1}$, and G_{m-1} for all essential values of m.

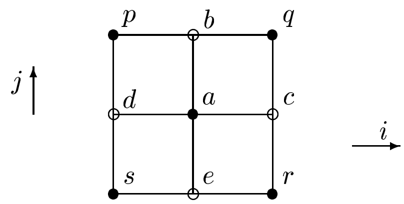


FIGURE 7

a_m	Q_m	$\widetilde{G}_{m-1}$	G_{m-1}
b	$\alpha x_p x_q$	$\alpha x_c x_d + \alpha\beta \frac{x_p x_q}{x_b} x_e$	$x_b x_c x_d + \beta x_e x_p x_q$
c	$\beta x_q x_r$	$\beta \frac{x_q x_r}{x_c} x_b x_d + \beta x_e x_p x_q$	$x_b x_d x_r + x_c x_e x_p$
d	$\beta x_p x_s$	$\beta \frac{x_p x_s}{x_d} x_b x_r + x_c x_e x_p$	$\beta x_b x_r x_s + x_c x_d x_e$
e	$\alpha x_r x_s$	$\beta x_b x_r x_s + \alpha \frac{x_r x_s}{x_e} x_c x_d$	$\beta x_b x_e + \alpha x_c x_d$

We see that $G_0 = G_{e-1} = P_a$, completing the proof. □

Proof of Theorem 1.8. The proof mimics the above proof of Theorem 1.4. To prove the Laurentness of an element y_N of the sequence (y_m) satisfying (1.10), we define the array $(z_{ijk})_{(i,j,k)\in\mathcal{H}}$ by setting $z_{ijk} = y_{N+\ell(i,j,k)}$, where $\ell(i,j,k) = n\frac{i+j+k}{2} - pi - qj$. The indexing set $\mathcal{H}$ is now given by

$$\mathcal{H} = \mathcal{H}(N) = \{(i,j,k) \in \mathbb{Z}^3 \,:\, N + \ell(i,j,k) \geq 0\}\,.$$

Then (1.10) implies that the z_{ijk} satisfy the octahedron recurrence (1.5). It is easy to check that $\mathcal{H}$ satisfies the conditions (1.7)–(1.8). Thus Theorem 1.6 applies to (z_{ijk}), with $H_{\rm init} = \{(a,b,c) \in L \,:\, 0 \leq N + \ell(a,b,c) < n\}$, and the theorem follows. □

We conclude this section by a couple of examples in which the Laurent phenomenon is established by the same technique as above. In each case, we provide:

- a picture of the equivalence classes "surrounding" a given class a, which plays the role of (4.7) in Example 4.1;
- the initialization polynomial $G_{N-1} = P_a$;
- a table showing a_m, Q_m, $\widetilde{G}_{m-1}$, and G_{m-1} for all essential values of m.

Example 4.3. *(Frieze patterns)* The generalized frieze pattern recurrence (cf., e.g., [3, 11]) is

$$y_{ij} y_{i-1,j-1} = \varepsilon\, y_{i,j-1} y_{i-1,j} + \beta\,, \tag{4.13}$$

where $\varepsilon \in \{1,-1\}$. To prove Laurentness (over $\mathbb{Z}[\beta]$), refer to Figure 8. Then $P_a = \varepsilon\, x_b\, x_c + \beta$, and the essential steps are:

a_m	Q_m	$\widetilde{G}_{m-1}$	G_{m-1}
b	β	$\frac{\varepsilon\,\beta\, x_c}{x_b} + \beta$	$\varepsilon\, x_c + x_b$
c	β	$\frac{\varepsilon\,\beta}{x_c} + x_b$	$\beta + \varepsilon^{-1} x_b\, x_c$

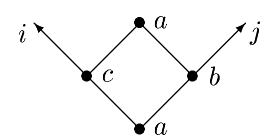


FIGURE 8

Example 4.4. *(Number walls)* Consider the 2-dimensional recurrence

$$y_{ij}y_{i,j-2} = y^p_{i-1,j-1}y^r_{i+1,j-1} + y^q_{i,j-1}\,, \tag{4.14}$$

where p, q, and r are nonnegative integers. To prove Laurentness, refer to Figure 9. Then $P_a = x_d^p x_b^r + x_c^q$, and the essential steps are:

a_m	Q_m	$\widetilde{G}_{m-1}$	G_{m-1}
b	x_f^q	$x_d^p\big(\frac{x_f^q}{x_b}\big)^r + x_c^q$	$x_d^p x_f^{qr} + x_c^q x_b^r$
c	$x_g^p x_f^r$	$x_d^p x_f^{qr} + \big(\frac{x_g^p x_f^r}{x_c}\big)^q x_b^r$	$x_d^p x_c^q + x_g^{pq} x_b^r$
d	x_g^q	$\big(\frac{x_g^q}{x_d}\big)^p x_c^q + x_g^{pq} x_b^r$	$x_c^q + x_b^r x_d^p$

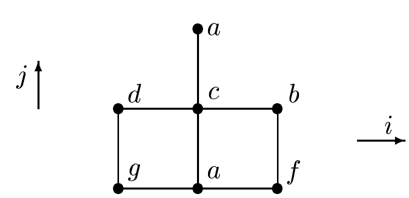


FIGURE 9

Remark 4.5. As pointed out by J. Propp, the Laurent phenomenon for certain special cases of Examples 4.3 and 4.4 can be obtained by specialization of Example 1.5.

5. Homogeneous exchange patterns

In this section, we deduce Theorem 1.10 and a number of similar results from the following corollary of Theorem 2.1.

Corollary 5.1. *Let $\mathbb{A}$ be a unique factorization domain. Assume that a collection of nonzero polynomials $P_1, \dots, P_n \in \mathbb{A}[x_1, \dots, x_n]$ satisfies the following conditions:*

(5.1) *Each P_k does not depend on x_k, and is not divisible by any x_i, $i \in [n]$.*

(5.2) *For any $i \neq j$, the polynomials $P_{ji} \stackrel{\text{def}}{=} (P_j)|_{x_i=0}$ and P_i are coprime.*

(5.3) *For any $i \neq j$, we have*

$$L \cdot P_{ji}^b \cdot P_i = P_i\Big|_{x_j \leftarrow \frac{P_{ji}}{x_j}} ,$$

where b is a nonnegative integer, and L is a Laurent monomial whose coefficient lies in $\mathbb{A}$ and is coprime with P_i.

Let us define the rational transformations F_i, $i \in [n]$, by

$$F_i : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, \frac{P_i}{x_i}, x_{i+1}, \dots, x_n).$$

Then any composition of the form $F_{i_1} \circ \dots \circ F_{i_m}$ is given by Laurent polynomials with coefficients in $\mathbb{A}$.

Proof. Let $\mathbb{T}_n$ denote a regular tree of degree n whose edges are labeled by elements of $[n]$ so that all edges incident to a given vertex have different labels. Assigning P_i as an exchange polynomial for every edge of $\mathbb{T}_n$ labeled by i, we obtain a "homogeneous" exchange pattern on $\mathbb{T}_n$ satisfying conditions (2.3)–(2.5) in Theorem 2.1. This implies the desired Laurentness. □

Example 5.2. Let $n \geq 3$ be an integer, and let P be a quadratic form given by

$$P(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 + \sum_{i<j} \alpha_{ij} x_i x_j .$$

Theorem 1.10 is a special case of Corollary 5.1 for $P_i = P\big|_{x_i=0}$ and $\mathbb{A} = \mathbb{Z}[\alpha_{ij} : i < j]$. Conditions (5.1)–(5.2) are clear. To verify (5.3), note that

$$P_i = P_{ji} + x_j^2 + x_j \Big(\sum_k \alpha_{kj} x_k + \sum_\ell \alpha_{j\ell} x_l \Big) ,$$

where k (resp. ℓ) runs over all indices such that $k \neq i$ and $k < j$ (resp. $\ell \neq i$ and $\ell > j$). It follows that

$$P_i\Big|_{x_j \leftarrow \frac{P_{ji}}{x_j}} = P_{ji} + \frac{P_{ji}^2}{x_j^2} + \frac{P_{ji}}{x_j} \Big(\sum_k \alpha_{kj} x_k + \sum_\ell \alpha_{j\ell} x_l \Big) = \frac{P_{ji}}{x_j^2} P_i ,$$

verifying (5.3).

In the remainder of this section, we list a few more applications of Corollary 5.1. In each case, the verification of its conditions is straightforward.

Example 5.3. Let P and Q be monic palindromic polynomials in one variable:

$$P(x) = (1 + x^d) + \alpha_1(x + x^{d-1}) + \alpha_2(x^2 + x^{d-2}) + \dots ;$$

$$Q(x) = (1 + x^e) + \beta_1(x + x^{e-1}) + \beta_2(x^2 + x^{e-2}) + \dots .$$

Then every member of the sequence $y_0, y_1, \ldots$ defined by the recurrence

$$y_k = \begin{cases} \dfrac{\mu^2 P(y_{k-1}/\lambda)}{y_{k-2}} & \text{if } k \text{ is odd;} \\ \dfrac{\lambda^2 Q(y_{k-1}/\mu)}{y_{k-2}} & \text{if } k \text{ is even} \end{cases}$$

is a Laurent polynomial in y_0 and y_1 with coefficients in $\mathbb{A} = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, \alpha_i, \beta_i]$. This follows from Corollary 5.1 with $n = 2$, $P_1 = \mu^2 P(x_2/\lambda)$, and $P_2 = \lambda^2 Q(x_1/\mu)$.

Example 5.4. Consider the sequence $y_0, y_1, \ldots$ defined by the recurrence

$$y_k = \frac{y_{k-1}^2 + c y_{k-1} + d}{y_{k-2}}. \tag{5.4}$$

Every term of this sequence is a Laurent polynomial in y_0 and y_1 with coefficients in $\mathbb{Z}[c, d]$.

Example 5.5. Define the rational transformations F_1, F_2, F_3 by

$$\begin{aligned} F_1 &: (x_1, x_2, x_3) \mapsto \left(\frac{x_2 + x_3^2 + x_2^2 x_3}{x_1}, \quad x_2, \quad x_3 \right), \\ F_2 &: (x_1, x_2, x_3) \mapsto \left(x_1, \quad \frac{x_1 + x_3}{x_2}, \quad x_3 \right), \\ F_3 &: (x_1, x_2, x_3) \mapsto \left(x_1, \quad x_2, \quad \frac{x_2 + x_1^2 + x_2^2 x_1}{x_3} \right). \end{aligned} \tag{5.5}$$

Then any composition $F_{i_1} \circ F_{i_2} \circ \cdots$ is given by $(x_1, x_2, x_3) \mapsto (G_1, G_2, G_3)$, where G_1, G_2, G_3 are Laurent polynomials in x_1, x_2, x_3 over $\mathbb{Z}$.

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