$A = \{a_{i,j}\}, \quad i,j \geq 0$

$B = \{b_{i,j}\}, \quad i,j \geq 0$

$C = \{c_{i,j}\}, \quad i,j \geq 0, \quad \text{where} \quad a_{i+1,j+1} \cdot c_{i,j} = b_{i,j}b_{i+1,j+1} - b_{i,j+1}b_{i+1,j}$

$D = \{d_{i,j}\}, \quad i,j \geq 0, \quad \text{where} \quad b_{i+1,j+1} \cdot d_{i,j} = c_{i,j}c_{i+1,j+1} - c_{i,j+1}c_{i+1,j}$

$E = \{e_{i,j}\}, \quad i,j \geq 0, \quad \text{where} \quad c_{i+1,j+1} \cdot e_{i,j} = d_{i,j}d_{i+1,j+1} - d_{i,j+1}d_{i+1,j}$

$F = \ldots, \quad i,j \geq 0$ etc

Hales' Identity is: $e_{0,0}b_{1,1}b_{2,2} + a_{2,2}d_{0,1}d_{1,0} = \det \mathcal{C}, \quad \text{where} \quad \mathcal{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} \\ c_{1,0} & c_{1,1} & c_{1,2} \\ c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}$

To establish this formula, start with:

$e_{0,0}c_{1,1} = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$

$$e_{0,0}c_{1,1}b_{1,1}b_{2,2}b_{1,2}b_{2,1} = (c_{0,0}c_{1,1} - c_{0,1}c_{1,0})(c_{1,1}c_{2,2} - c_{1,2}c_{2,1})b_{1,2}b_{2,1} - (c_{0,1}c_{1,2} - c_{0,2}c_{1,1})(c_{1,0}c_{2,1} - c_{1,1}c_{2,0})b_{1,1}b_{2,2}$$

$$= (c_{0,0}c_{1,1} - c_{0,1}c_{1,0})(c_{1,1}c_{2,2} - c_{1,2}c_{2,1})b_{1,2}b_{2,1} - (c_{0,1}c_{1,2} - c_{0,2}c_{1,1})(c_{1,0}c_{2,1} - c_{1,1}c_{2,0})b_{1,2}b_{2,1} + (c_{0,1}c_{1,2} - c_{0,2}c_{1,1})(c_{1,0}c_{2,1} - c_{1,1}c_{2,0})a_{2,2}c_{1,1}$$

$$= (c_{0,0}c_{1,1}c_{1,1}c_{2,2} - c_{0,0}c_{1,1}c_{1,2}c_{2,1} - c_{0,1}c_{1,0}c_{1,1}c_{2,2} + c_{0,1}c_{1,0}c_{1,2}c_{2,1} + -c_{0,1}c_{1,2}c_{1,0}c_{2,1} + c_{0,1}c_{1,2}c_{1,1}c_{2,0} + c_{0,2}c_{1,1}c_{1,0}c_{2,1} - c_{0,2}c_{1,1}c_{1,1}c_{2,0})b_{1,2}b_{2,1} - a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}c_{1,1}$$

$$= \det \mathcal{C} \cdot b_{1,2}b_{2,1}c_{1,1} - a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}c_{1,1}$$

Dividing by $c_{1,1}b_{1,2}b_{2,1}$ gives Hales' formula.

There is a similar formula with subscripts interchanged: $e_{0,0}b_{1,2}b_{2,1} + a_{2,2}d_{0,0}d_{1,1} = \det \mathcal{C}$

There is also a similar formula with the octahedron axis $A_{2,2} - C_{1,1} - E_{0,0}$ replaced by the axis $C_{0,0} - C_{1,1} - C_{2,2}$, and another one with axis $C_{0,2} - C_{1,1} - C_{2,0}$.

The axis $C_{0,2} - C_{1,1} - C_{2,0}$ is perpendicular to the plane $\mathcal{A} = \begin{bmatrix} a_{2,2} & b_{2,2} & c_{2,2} \\ b_{1,1} & c_{1,1} & d_{1,1} \\ c_{0,0} & d_{0,0} & e_{0,0} \end{bmatrix}$

$$\begin{aligned} \det \mathcal{A} &= c_{0,2}b_{2,1}d_{1,0} + c_{2,0}c_{0,1}c_{1,2} = c_{2,0}b_{1,2}d_{0,1} + c_{0,2}c_{1,0}c_{2,1} \\ &= -c_{0,2}c_{2,0}c_{1,1} + c_{0,2}c_{1,0}c_{2,1} + c_{2,0}c_{0,1}c_{1,2} \end{aligned}$$

The axis $C_{0,0} - C_{1,1} - C_{2,2}$ is perpendicular to the plane $\mathcal{B} = \begin{bmatrix} a_{2,2} & b_{1,2} & c_{0,2} \\ b_{2,1} & c_{1,1} & d_{0,1} \\ c_{2,0} & d_{1,0} & e_{0,0} \end{bmatrix}$

$$\begin{aligned} \det \mathcal{B} &= c_{0,0}b_{2,2}d_{1,1} + c_{2,2}c_{0,1}c_{1,0} = c_{0,0}c_{1,2}c_{2,1} + c_{2,2}b_{1,1}d_{0,0} \\ &= -c_{0,0}c_{1,1}c_{2,2} + c_{0,0}c_{1,2}c_{2.1} + c_{2,2}c_{0,1}c_{1,0} \end{aligned}$$

$$\begin{aligned} \det \mathcal{C} &= e_{0,0}b_{1,1}b_{2,2} + a_{2,2}d_{0,1}d_{1,0} \\ &= e_{0,0}b_{1,2}b_{2,1} + a_{2,2}d_{0,0}d_{1,1} \end{aligned} \quad \text{where} \quad \mathcal{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & c_{0,2} \\ c_{1,0} & c_{1,1} & c_{1,2} \\ c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}$$

$\det \mathcal{A} - \det \mathcal{B} = \det \mathcal{C}$

Assuming an error of Δ $(= \Delta c_{1,1})$ at $C_{1,1}$ The calculation of the error at $E_{0,0}$ is:

$$\Delta e_{0,0} = \Big(\frac{c_{0,0}d_{1,1}}{b_{1,1}c_{1,1}} + \frac{c_{2,2}d_{0,0}}{b_{2,2}c_{1,1}} + \frac{c_{0,2}d_{1,0}}{b_{1,2}c_{1,1}} + \frac{c_{2,0}d_{0,1}}{b_{2,1}c_{1,1}} - \frac{e_{0,0}}{c_{1,1}}\Big)\Delta \qquad \text{Let } \delta e_{0,0} = \frac{\Delta e_{0,0}}{\Delta}.$$

$$(\delta e_{0,0}) \cdot b_{1,1}b_{2,2}b_{1,2}b_{2,1}c_{1,1} = c_{0,0}d_{1,1}b_{2,2}b_{1,2}b_{2,1} + c_{2,2}d_{0,0}b_{1,1}b_{1,2}b_{2,1} +$$

$$c_{0,2}d_{1,0}b_{2,1}b_{1,1}b_{2,2} + c_{2,0}d_{0,1}b_{1,2}b_{1,1}b_{2,2} - e_{0,0}b_{1,1}b_{2,2}b_{1,2}b_{2,1}$$

$$= \Big(c_{0,0}(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + c_{2,2}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0})\Big) \cdot b_{1,2}b_{2,1} +$$

$$\Big(c_{0,2}d_{1,0}b_{2,1} + c_{2,0}d_{0,1}b_{1,2}\Big) \cdot \Big(c_{1,1}a_{2,2} + b_{1,2}b_{2,1}\Big) - \det\mathcal{C} \cdot b_{1,2}b_{2,1} + a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}$$

$$= \Big(c_{0,0}(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + c_{2,2}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}) +$$

$$c_{0,2}(c_{1,0}c_{2,1} - c_{1,1}c_{2,0}) + c_{2,0}(c_{0,1}c_{1,2} - c_{0,2}c_{1,1})\Big) \cdot b_{1,2}b_{2,1} - \det\mathcal{C} \cdot b_{1,2}b_{2,1} +$$

$$(c_{0,2}d_{1,0}b_{2,1} + c_{2,0}d_{0,1}b_{1,2})c_{1,1}a_{2,2} + a_{2,2}d_{0,1}d_{1,0}b_{1,2}b_{2,1}$$

$$= (c_{0,0}c_{2,2} - c_{0,2}c_{2,0})b_{1,2}b_{2,1}c_{1,1} + c_{2,0}d_{0,1}b_{1,2}a_{2,2}c_{1,1} + c_{0,2}d_{1,0}b_{2,1}a_{2,2}c_{1,1} + d_{0,1}d_{1,0}a_{2,2}b_{1,2}b_{2,1}$$

Then $\quad \Delta e_{0,0} = \Big[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{b_{1,1}b_{2,2}} + \frac{c_{2,0}d_{0,1}a_{2,2}}{b_{2,1}b_{1,1}b_{2,2}} + \frac{c_{0,2}d_{1,0}a_{2,2}}{b_{1,2}b_{1,1}b_{2,2}} + \frac{a_{2,2}d_{1,0}d_{0,1}}{c_{1,1}b_{1,1}b_{2,2}}\Big] \cdot \Delta$

Start with $\quad \Delta c_{1,1} = \Delta = \frac{b_{1,1}b_{2,2}}{a_{2,2}} \cdot \epsilon, \quad$ then

$$\Delta e_{0,0} = \Big[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{a_{2,2}} + \frac{c_{2,0}d_{0,1}}{b_{2,1}} + \frac{c_{0,2}d_{1,0}}{b_{1,2}} + \frac{d_{1,0}d_{0,1}}{c_{1,1}}\Big] \cdot \epsilon$$

This guarantees that $\Delta e_{0,0}$ has an expression as a sum of terms, each of which has valuation $\geq p - q$, where p is the precision and $q =$ max valuation of $b_{1,1}, b_{2,2}, b_{1,2}, b_{2,1}, c_{1,1}$

The important thing is that $\Delta e_{0,0} \cdot (b_{1,1}b_{2,2}b_{1,2}b_{2,1}c_{1,1})$ belongs to the ideal generated by

$$b_{1,2}b_{2,1}c_{1,1},\ a_{2,2}b_{1,2}c_{1,1},\ a_{2,2}b_{2,1}c_{1,1},\ a_{2,2}b_{1,2}b_{2,1},\ b_{1,1}b_{2,2}c_{1,1},\ a_{2,2}b_{1,1}c_{1,1},\ a_{2,2}b_{2,2}c_{1,1},\ a_{2,2}b_{1,1}b_{2,2}$$

There is a similar formula with denominators $b_{1,1}$ and $b_{2,2}$ instead of $b_{1,2}$ and $b_{2,1}$:

$$\Delta e_{0,0} = \Big[\frac{c_{0,0}c_{2,2} - c_{0,2}c_{2,0}}{a_{2,2}} - \frac{c_{0,0}d_{1,1}}{b_{1,1}} - \frac{c_{2,2}d_{1,1}}{b_{2,2}} + \frac{d_{0,0}d_{1,1}}{c_{1,1}}\Big] \cdot \epsilon$$