06/23/2014 draft

Let $\pi = \{x_{i,j}\}$, be the starting plane of indeterminates, and let $R = Z[x_{i,j}]$ be the polynomial ring with these indeterminates. For each $n \geq 1$, let $x_{i,j}(n)$ be the $n \times n$ minor with $x_{i,j}$ in the upper left corner. The $x_{i,j}(n)$ satisfy the Dodgson recursion: For $n \geq 3$,

$$x_{i+1,j+1}(n-2) \cdot x_{i,j}(n) = x_{i,j}(n-1) \cdot x_{i+1,j+1}(n-1) - x_{i,j+1}(n-1) \cdot x_{i+1,j}(n-1)$$

The position of $x_{i,j}(n)$ will be designated (i,j,n). Suppose for each i,j and each $n \geq 3$, a change is made in the computation of $x_{i,j}(n)$; the computed expressions will be denoted $X_{i,j}(n)$, i.e. upper case letters. The computation of $X_{i,j}(n)$ has the form:

$$X_{i+1,j+1}(n-2) \cdot X_{i,j}(n) = X_{i,j}(n-1) \cdot X_{i+1,j+1}(n-1) \cdot (1+\epsilon) - X_{i,j+1}(n-1) \cdot X_{i+1,j}(n-1)$$

In the following calculations of $X_{i,j}(n)$, for each n we need to consider only a finite number of the $x_{i,j}$, so we may assume thaat R is finitely generated. We need to construct a ring A in which the computed expressions lie. For each $n \geq 3$, and each position $\alpha = (i,j,n))$, there will be a new indeterminate ϵ_α. For each such ϵ_α, there are four "denominators", as follows:

$$\begin{aligned} y_{2,2} &= X_{i+1,j+1}(n-2) \\ y_{2,1} &= X_{i+1,j}(n-1) \\ y_{1,2} &= X_{i,j+1}(n-1) \\ y_{1,1} &= X_{i,j}(n) \end{aligned}$$

Finally, let
$$A = R[\frac{\epsilon_\alpha}{y}]$$

where α varies over all $\alpha = (i,j,n))$, and the y varies over the four $y's$ just defined.

Conjecture All the computed expressions $X_{i,j}(n)$ are in A.

The calculations and discussion below establish this conjecture for a special case (two steps). That is, assuming a chnge is made at level n, we show that the changed expressions at levels $n+1$ and $n+2$ are in A.

Suppose we pass levels $n-2$, $n-1$, and the first change is made at level n. To simplify notation, the variables at level $n-2$ will be denoted $\{a_{i,j}\}$. Similarly for for $\{b_{i,j}\}$, $\{c_{i,j}\}$, $\{d_{i,j}\}$, and $\{e_{i,j}\}$ at levels $n-1$, n, $n+1$ and $n+2$.

Let $R = Z[x_{i,j}, \delta]$. Suppose that the computation of $C_{1,1,}$ is made as:

$$\begin{aligned} a_{2,2}C_{1,1} &= (1+\epsilon)b_{1,1}b_{2,2} - b_{1,2}b_{2,1} \\ &= a_{2,2}c_{1,1} + b_{1,1}b_{2,2}\epsilon \\ C_{1,1} &= c_{1,1} + b_{1,1}b_{2,2}\frac{\epsilon}{a_{2,2}} \end{aligned}$$

(Step 1) The elements $D_{i,j}$ are all in A.

$$\begin{aligned} b_{1,1}D_{0,0} &= c_{0,0}C_{1,1} - c_{0,1}c_{1,0} \\ &= c_{0,0}(c_{1,1} + b_{1,1}b_{2,2}\frac{\epsilon}{a_{2,2}}) - c_{0,1}c_{1,0} \\ D_{0,0} &= d_{0,0} + c_{0,0}b_{2,2}\frac{\epsilon}{a_{2,2}} \end{aligned}$$

Similarly,

$$\begin{aligned} D_{1,1} &= d_{1,1} + c_{2,2}b_{1,1}\frac{\epsilon}{a_{2,2}} \\ D_{1,2} &= d_{1,2} - c_{0,2}\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} \\ D_{2,1} &= d_{1,1} + c_{2,0}\frac{b_{1,1}b_{2,2}\epsilon}{b_{2,1}a_{2,2}} \end{aligned}$$

It would seem that $D_{0,1}$ involves a fraction that is not in A, because $\frac{\epsilon}{b_{1,2}a_{2,2}}$ is not in A. **BUT** $\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}}$ **IS** in A. The ideal $(a_{2,2}, b_{1,2})$ does not contain 1, but it does contain $b_{1,1}b_{2,2}$. This follows from the identity

$$b_{1,1}b_{2,2} = a_{2,2}c_{1,1} + b_{1,2}b_{2,1}$$

This shows that

$$\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}$$

Thus

$$D_{0,1} = d_{0,1} - c_{0,2}c_{1,1}\frac{\epsilon}{b_{1,2}} - c_{0,2}b_{2,1}\frac{\epsilon}{a_{2,2}}$$

Similarly,

$$D_{1,0} = d_{1,0} - c_{2,0}c_{1,1}\frac{\epsilon}{b_{2,1}} - c_{2,0}b_{1,2}\frac{\epsilon}{a_{2,2}}$$

We see that $D_{0,1}$ and $D_{1,0}$ are A.

For convenience,of notation, let $F = b_{1,1}b_{2,2}b_{1,2}b_{2,1}$, and $\epsilon = b_{1,2}b_{2,1}a_{2,2}\delta$. Thus $b_{1,1}b_{2,2}\epsilon = a_{2,2}F\delta$.

$$\begin{aligned}
C_{1,1} &= c_{1,1} + FC_{1,1}\delta \\
D_{1,1} &= d_{1,1} + c_{2,2}\frac{F}{b_{2,2}}\delta \\
D_{0,1} &= d_{0,1} - c_{0,2}\frac{F}{b_{1,2}}\delta \\
D_{1,0} &= d_{1,0} - c_{2,0}\frac{F}{b_{2,1}}\delta \\
D_{0,0} &= d_{0,0} + c_{0,0}\frac{F}{b_{1,1}}\delta
\end{aligned}$$

(2) Step 2 $E_{0,0}$ is in A

Proof: Let $Y = D_{0,0}D_{1,1} - D_{0,1}D_{1,0}$. Thus $E_{0,0} = \dfrac{Y}{C_{1,1}}$. Multiply Y by $b_{1,1}$, to get

$$\begin{aligned}
b_{1,1}Y &= D_{1,1}\big(c_{0,0}C_{1,1} - c_{0,1}c_{1,0}\big) - b_{1,1}D_{0,1}D_{1,0} \\
&= C_{1,1}X_1 - U
\end{aligned}$$

where $X_1 = D_{1,1}c_{0,0}$, and $U = c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}D_{1,0}$.

$$\begin{aligned}
b_{2,1}U &= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{2,1}b_{1,1}D_{0,1}D_{1,0} \\
&= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}\big(c_{1,0}c_{2,1} - C_{1,1}c_{2,0}\big) \\
&= c_{1,0}V - C_{1,1}X_2
\end{aligned}$$

where $X_2 = b_{1,1}D_{0,1}c_{2,0}$ and $V = b_{2,1}D_{1,1}c_{0,1} + b_{1,1}D_{0,1}c_{2,1}$

$$
\begin{aligned}
b_{2,2}V &= b_{2,2}b_{2,1}D_{1,1}c_{0,1} + b_{2,2}b_{1,1}D_{0,1}c_{2,1} \\
&= b_{2,1}c_{0,1}\big(C_{1,1}c_{2,2} - c_{1,2}c_{2,1}\big) + b_{2,2}b_{1,1}D_{0,1}c_{2,1} \\
&= C_{1,1}X_3 + c_{2,1}W
\end{aligned}
$$

where $X_3 = b_{2,1}c_{0,1}c_{2,2}$ and $W = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}D_{0,1}$

$$
\begin{aligned}
b_{1,2}W &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}b_{1,2}D_{0,1} \\
&= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}\big(c_{0,1}c_{1,2} - c_{0,2}C_{1,1})\big) \\
&= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2}) \\
&= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(a_{2,2}C_{1,1} - b_{1,1}b_{2,2}\epsilon) \\
&= X_4C_{1,1} - c_{0,1}c_{1,2}a_{2,2}F\delta
\end{aligned}
$$

where

$$
\begin{aligned}
X_4 &= a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2} \\
&= a_{2,2}b_{1,2}d_{0,1} - b_{1,2}b_{2,1}c_{0,2}
\end{aligned}
$$

A calculation (reversing the steps above), shows that

$$\begin{aligned}
FY &= b_{1,2}b_{2,2}b_{2,1}C_{1,1}X_1 \\
&- b_{1,2}b_{2,2}C_{1,1}X_2 \\
&- b_{1,2}c_{1,0}C_{1,1}X_3 \\
&- c_{1,0}c_{2,1}C_{1,1}X_4 + c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta \\
\\
&= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}D_{1,1}c_{0,0} \\
&- b_{1,2}b_{2,2}b_{1,1}c_{2,0}D_{0,1} \\
&- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\
&- c_{1,0}c_{2,1}X_4\big) + c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta \\
\\
&= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}\big(d_{1,1} + c_{2,2}\frac{F}{b_{2,2}}\delta\big) \\
&- b_{1,2}b_{2,2}b_{1,1}c_{2,0}\big(d_{0,1} - c_{0,2}\frac{F}{b_{1,2}}\delta\big) \\
&- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2}\big) \\
&+ c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta
\end{aligned}$$

$$\begin{aligned}
FY &= C_{1,1}\big(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \\
&- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}\big) \\
&+ b_{1,2}b_{2,1}c_{0,0}c_{2,2}F\delta \\
&- b_{1,1}b_{2,2}c_{2,0}c_{0,2}F\delta\big) \\
&+ c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta \\
&= C_{1,1}p(x_{i,j}) + FZ
\end{aligned}$$

where

$$\begin{aligned} p(x) &= b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\ &+ c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \\ Z &= C_{1,1}b_{1,2}b_{2,1}c_{0,0}c_{2,2}\delta \\ &- C_{1,1}b_{1,2}b_{2,1}c_{2,0}c_{0,2}\delta \\ &+ c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}\delta - C_{1,1}a_{2,2}c_{1,1}c_{2,0}c_{0,2}\delta \end{aligned}$$

(1) If we set δ to 0, then Y becomes $y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$, and we have

$$b_{1,1}b_{2,2}b_{1,2}b_{2,1}y = c_{1,1}p(x)$$

Each of these expressions $(b_{1,1}b_{2,2}b_{1,2}b_{2,1}, y, c_{1,1}, p(x))$ is a polynomial in the original ring, $Z[x_{i,j}]$, and $c_{1,1}$ is indecomposable. Therefore

$$\frac{p(x)}{F} = \frac{y}{c_{1,1}} = e_{0,0}$$

(2) The expression for Z must now be divided by $C_{1,1}$ The first and second terms are easy:

$$\begin{aligned} b_{1,2}b_{2,1}c_{0,0}c_{2,2}\delta &= c_{0,0}c_{2,2}\frac{\epsilon}{a_{2,2}} \\ b_{1,2}b_{2,1}c_{2,0}c_{2,0}\delta &= c_{2,0}c_{0,2}\frac{\epsilon}{a_{2,2}} \end{aligned}$$

(3) Let $P = \Big(c_{1,0}c_{2,1}c_{0,1}c_{1,2} - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}\delta$. Using

$$c_{1,0}c_{2,1} = b_{2,1}D_{1,0} + c_{2,0}C_{1,1}$$

$$c_{0,1}c_{1,2} = b_{1,2}D_{0,1} + c_{0,2}C_{1,1},$$

we have

$$
\begin{aligned}
P &= \Big((b_{2,1}D_{1,0} + c_{2,0}\cdot(b_{1,2}D_{0,1} + c_{0,2}C_{1,1}) - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}C_{1,1}\delta \\
&= b_{2,1}b_{1,2}D_{1,0}D_{0,1}a_{2,2}\delta \\
&+ b_{2,1}D_{1,0}c_{0,2}C_{1,1}a_{2,2}\delta \\
&+ c_{2,0}b_{1,2}D_{0,1}C_{1,1}a_{2,2}\delta \\
&+ \Big(c_{2,0}C_{1,1}c_{0,2}C_{1,1} - C_{1,1}c_{1,1}c_{2,0}c_{0,2}\Big)a_{2,2}\delta \\
&= D_{1,0}D_{0,1}\epsilon \\
&+ D_{1,0}c_{0,2}C_{1,1}\frac{\epsilon}{b_{1,2}} \\
&+ c_{2,0}D_{0,1}C_{1,1}\frac{\epsilon}{b_{2,1}} \\
&+ C_{1,1}c_{2,0}c_{0,2}\cdot(C_{1,1} - c_{1,1})\cdot\frac{\epsilon}{b_{1,2}b_{2,1}}
\end{aligned}
$$

(3a) After dividing by $C_{1,1}$, the first term becomes $D_{1,0}D_{0,1}\frac{\epsilon}{C_{1,1}}$, which is in A. (After expanding the definitions of $D_{1,0}, D_{0,1}$, this will involve some quadratic and cubic terms in ϵ.)

(3b) After dividing by $C_{1,1}$, the second term becomes $D_{1,0}c_{0,2}\frac{\epsilon}{b_{1,2}}$, which is in A.

(3c) After dividing by $C_{1,1}$, the third term becomes $c_{2,0}D_{0,1}\frac{\epsilon}{b_{2,1}}$, which is in A.

(3d) After dividing by $C_{1,1}$, and using $C_{1,1} - c_{1,1} = \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}}$, the fourth term becomes

$$c_{2,0}c_{0,2}\cdot\frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}}\cdot\frac{\epsilon}{b_{1,2}b_{2,1}}.$$

Using the identity:

$$\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}$$

this becomes

$$c_{2,0}c_{0,2}\cdot\left(c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}\right)\frac{\epsilon}{b_{2,1}}$$

which is in A. (Some terms quadratic in ϵ appear.)

****

To summarize and clarify the situation, this calculation shows that

$$FY = C_{1,1}p(x) + C_{1,1}FZ_1 + FD_{1,0}D_{0,1} \cdot \epsilon,$$

where

(1) $p(x)$ is the polynomial in $Z[x_{i,j}]$ that results if ϵ is set to 0, and $\dfrac{p(x)}{F} = e_{0,0}$.

(2) $Z_1 = \dfrac{Z}{C_{1,1}}$ is demonstrably in $R[\dfrac{\epsilon}{a_{2,2}}, \dfrac{\epsilon}{b_{1,2}}, \dfrac{\epsilon}{b_{2,2}}]$

(3) $\dfrac{\epsilon}{C_{1,1}}$ is the fraction adjoined to A at this stage. Therefore,
$E_{0,0} = \dfrac{Y}{C_{1,1}} = e_{0,0} + Z_1 + D_{1,0}D_{0,1}\dfrac{\epsilon}{C_{1,1}}$ is in A.

Step 3 $E_{0,1}$ is in A

Proof: Let $Y = D_{0,1}D_{1,2} - D_{0,2}D_{1,1}$. We must show that Y is divisible by $C_{1,2}$, and thus $E_{0,1} = \dfrac{Y}{C_{1,2}}$. Multiply Y by $b_{1,2}$, to get

$$\begin{aligned} b_{1,2}Y &= D_{1,2}\big(c_{0,1}c_{1,2} - c_{0,2}C_{1,1}\big) - b_{1,2}d_{0,2}D_{1,1} \\ &= c_{1,2}X_1 - U \end{aligned}$$

where $X_1 = d_{1,2}c_{0,1}$, and $U = c_{0,2}C_{1,1}d_{1,2} + b_{1,2}d_{0,2}D_{1,1}$.

$$\begin{aligned} b_{2,2}U &= b_{2,2}c_{0,2}C_{1,1}d_{1,2} + b_{2,2}b_{1,2}d_{0,2}D_{1,1} \\ &= b_{2,2}c_{0,2}C_{1,1}d_{1,2} + b_{1,2}d_{0,2}\big(C_{1,1}c_{2,2} - c_{1,2}c_{2,1}\big) \\ &= C_{1,1}V - c_{1,2}X_2 \end{aligned}$$

where $X_2 = b_{1,2}d_{0,2}c_{2,1}$ and $V = b_{2,2}c_{0,2}d_{1,2} + b_{1,2}d_{0,2}c_{2,2}$

$$\begin{aligned} b_{2,3}V &= b_{2,3}b_{2,2}d_{1,2}c_{0,2} + b_{2,3}b_{1,2}d_{0,2}c_{2,2} \\ &= b_{2,2}c_{0,2}\big(c_{1,2}c_{2,3} - c_{1,3}c_{2,2}\big) + b_{2,3}b_{1,2}d_{0,2}c_{2,2} \\ &= c_{1,2}X_3 + c_{2,2}W \end{aligned}$$

where $X_3 = b_{2,2}c_{0,2}c_{2,3}$ and $W = -b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}d_{0,2}$

$$\begin{aligned} b_{1,3}W &= -b_{1,3}b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}b_{1,3}d_{0,2} \\ &= -b_{1,3}b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}\big(c_{0,2}c_{1,3} - c_{0,3}c_{1,2})\big) \\ &= -b_{1,2}b_{2,3}c_{0,3}c_{1,2} + c_{0,2}c_{1,3}(-b_{1,3}b_{2,2} + b_{1,2}b_{2,3}) \\ &= -b_{1,2}b_{2,3}c_{0,3}c_{1,2} + c_{0,2}c_{1,3}(a_{2,3}c_{1,2}) \\ &= X_4c_{1,2} \end{aligned}$$

where

$$\begin{aligned} X_4 &= a_{2,3}c_{0,2}c_{1,3} - b_{1,2}b_{2,3}c_{0,3} \\ &= a_{2,3}b_{1,3}d_{0,2} - b_{1,3}b_{2,2}c_{0,3} \\ W &= (a_{2,3}d_{0,2} - b_{2,2}c_{0,3})c_{1,2} = X_5c_{1,2} \end{aligned}$$

$$\begin{aligned} b_{2,3}V &= c_{1,2}(X_3 + c_{2,2}X_5) \\ &= c_{1,2}\big(b_{2,2}c_{0,2}c_{2,3} + c_{2,2}(a_{2,3}d_{0,2} - b_{2,2}c_{0,3})\big) \end{aligned}$$

$$
\begin{aligned}
b_{1,3}b_{2,3}b_{2,2}b_{1,2}Y &= b_{1,3}b_{2,3}b_{2,2}C_{1,2}X_1 \\
&- b_{1,3}b_{2,3}C_{1,2}X_2 \\
&- b_{1,3}c_{1,1}C_{1,2}X_3 \\
&- c_{1,1}c_{2,2}C_{1,2}X_4 \\
\\
&= C_{1,2}\big(b_{1,3}b_{2,3}b_{2,2}D_{1,2}c_{0,1} \\
&- b_{1,3}b_{2,3}b_{1,2}c_{2,1}D_{0,2} \\
&- b_{1,3}c_{1,1}b_{2,2}c_{0,2}c_{2,3} \\
&- c_{1,1}c_{2,2}X_4\big)
\end{aligned}
$$

****

Lemma Let R be a UFD, and A, B, C, and D be four planes of elements of R. Assume that for every Sylvester diamond, the Sylvester recurrence is true. That is,

$$
\begin{aligned}
a_{i+1,j+1}c_{i,j} &= b_{i,j}b_{i+1,j+1} - b_{i,j+1}b_{i+1,j} \\
b_{i+1,j+1}d_{i,j} &= c_{i,j}c_{i+1,j+1} - c_{i,j+1}c_{i+1,j}
\end{aligned}
$$

Then $d_{i,j} \cdot d_{i+1,j+1} - d_{i,j+1} \cdot d_{i+1,j}$ is divisible by $c_{i+1,j+1}$ in R.

Proof: By a shift of indices, it is sufficient to prove the assertion when $i, j = 0, 0$. That is, let $Y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$. We must show that Y is divisible by $c_{1,1}$, and that the quotient is in R. Multiply Y by $b_{1,1}$, to get

$$
\begin{aligned}
b_{1,1}Y &= d_{1,1}\big(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}\big) - b_{1,1}d_{0,1}d_{1,0} \\
&= c_{1,1}X_1 - U
\end{aligned}
$$

where $X_1 = d_{1,1}c_{0,0}$, and $U = c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}d_{1,0}$.

$$
\begin{aligned}
b_{2,1}U &= b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{2,1}b_{1,1}d_{0,1}d_{1,0} \\
&= b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}\big(c_{1,0}c_{2,1} - c_{1,1}c_{2,0}\big) \\
&= c_{1,0}V - c_{1,1}X_2
\end{aligned}
$$

where $X_2 = b_{1,1}d_{0,1}c_{2,0}$ and $V = b_{2,1}d_{1,1}c_{0,1} + b_{1,1}d_{0,1}c_{2,1}$

$$\begin{aligned} b_{2,2}V &= b_{2,2}b_{2,1}d_{1,1}c_{0,1} + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \\ &= b_{2,1}c_{0,1}\big(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}\big) + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \\ &= c_{1,1}X_3 + c_{2,1}W \end{aligned}$$

where $X_3 = b_{2,1}c_{0,1}c_{2,2}$ and $W = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}d_{0,1}$

$$\begin{aligned} b_{1,2}W &= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}\big(c_{0,1}c_{1,2} - c_{0,2}c_{1,1})\big) \\ &= -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2}) \\ &= -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(a_{2,2}c_{1,1}) \\ &= (a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2})c_{1,1} \\ &= (a_{2,2}d_{0,1} - b_{1,2}c_{0,2})b_{1,2}c_{1,1} \end{aligned}$$

Thus W is divisible by $c_{1,1}$, then V, then U, and finally Y is divisvble by $c_{1,1}$.

The quotient $\dfrac{Y}{c_{1,1}}$ will be called $e_{0,0}$. By a shift of indices, $e_{i,j}$ may be defined in R for all i, j. In this way, the planes are continued indefinitely $E, F, G, \ldots$ Similarly the planes can be continued in the other direction $B, A, Z, Y, \ldots$