Let $\pi = \{x_{i,j}\}$, be the starting plane of indeterminates, and let $R = \mathbb{Z}[x_{i,j}]$ be the polynomial ring with these indeterminates. For each $n \geq 1$, let $x_{i,j}(n)$ be the $n \times n$ minor with $x_{i,j}$ in the upper left corner. The $x_{i,j}(n)$ satisfy the Dodgson recursion:

For $n \geq 3$,

$$x_{i+1,j+1}(n-2) \cdot x_{i,j}(n) = x_{i,j}(n-1) \cdot x_{i+1,j+1}(n-1) - x_{i,j+1}(n-1) \cdot x_{i+1,j}(n-1)$$

The position of $x_{i,j}(n)$ will be designated $(i, j, n)$. Suppose for each $i, j$ and each $n \geq 3$, a change is made in the computation of $x_{i,j}(n)$; the computed expressions will be denoted $X_{i,j}(n)$, i.e. upper case letters. The computation of $X_{i,j}(n)$ has the form:

$$X_{i+1,j+1}(n-2) \cdot X_{i,j}(n) = X_{i,j}(n-1) \cdot X_{i+1,j+1}(n-1) \cdot (1+\epsilon) - X_{i,j+1}(n-1) \cdot X_{i+1,j}(n-1)$$

In the following calculations of $X_{i,j}(n)$, for each $n$ we need to consider only a finite number of the $x_{i,j}$, so we may assume that $R$ is finitely generated. We need to construct a ring $A$ in which the computed expressions lie. For each $n \geq 3$, and each position $\alpha = (i, j, n)$, there will be a new indeterminate $\epsilon_\alpha$. For each such $\epsilon_\alpha$, there are four “denominators”, as follows:

$$y_{2,2} = X_{i+1,j+1}(n-2)$$
$$y_{2,1} = X_{i+1,j}(n-1)$$
$$y_{1,2} = X_{i,j+1}(n-1)$$
$$y_{1,1} = X_{i,j}(n)$$

Finally, let

$$A = R[\frac{\epsilon_\alpha}{y}]$$

where $\alpha$ varies over all $\alpha = (i, j, n)$, and the $y$ varies over the four $y$'s just defined.

**Conjecture** All the computed expressions $X_{i,j}(n)$ are in $A$.

The calculations and discussion below establish this conjecture for a special case (two steps). That is, assuming a change is made at level $n$, we show that the changed expressions at levels $n+1$ and $n+2$ are in $A$.

Suppose we pass levels $n-2$, $n-1$, and the first change is made at level $n$. To simplify notation, the variables at level $n-2$ will be denoted $\{a_{i,j}\}$. Similarly for for $\{b_{i,j}\}$, $\{c_{i,j}\}$, $\{d_{i,j}\}$, and $\{e_{i,j}\}$ at levels $n-1$, $n$, $n+1$ and $n+2$.  

1
Let \( R = \mathbb{Z}[x_{i,j}, \delta] \). Suppose that the computation of \( C_{1,1} \) is made as:

\[
\begin{align*}
   a_{2,2}C_{1,1} &= (1 + \epsilon)b_{1,1}b_{2,2} - b_{1,2}b_{2,1} \\
   &= a_{2,2}c_{1,1} + b_{1,1}b_{2,2}\epsilon \\
   C_{1,1} &= c_{1,1} + b_{1,1}b_{2,2}\frac{\epsilon}{a_{2,2}}
\end{align*}
\]

\textbf{(Step 1)} The elements \( D_{i,j} \) are all in \( A \).

\[
\begin{align*}
   b_{1,1}D_{0,0} &= c_{0,0}C_{1,1} - c_{0,1}c_{1,0} \\
   &= c_{0,0}(c_{1,1} + b_{1,1}b_{2,2}\frac{\epsilon}{a_{2,2}}) - c_{0,1}c_{1,0} \\
   D_{0,0} &= d_{0,0} + c_{0,0}b_{2,2}\frac{\epsilon}{a_{2,2}}
\end{align*}
\]

Similarly,

\[
\begin{align*}
   D_{1,1} &= d_{1,1} + c_{2,2}b_{1,1}\frac{\epsilon}{a_{2,2}} \\
   D_{1,2} &= d_{1,2} - c_{0,2}\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} \\
   D_{2,1} &= d_{1,1} + c_{2,0}\frac{b_{1,1}b_{2,2}\epsilon}{b_{2,1}a_{2,2}}
\end{align*}
\]

It would seem that \( D_{0,1} \) involves a fraction that is not in \( A \), because \( \frac{\epsilon}{b_{1,2}a_{2,2}} \) is not in \( A \). \textbf{BUT} \( \frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} \) \textbf{IS} in \( A \). The ideal \((a_{2,2}, b_{1,2})\) does not contain 1, but it does contain \( b_{1,1}b_{2,2} \). This follows from the identity

\[
b_{1,1}b_{2,2} = a_{2,2}c_{1,1} + b_{1,2}b_{2,1}
\]

This shows that

\[
\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1}\frac{\epsilon}{b_{1,2}} + b_{2,1}\frac{\epsilon}{a_{2,2}}
\]

Thus

\[
D_{0,1} = d_{0,1} - c_{0,2}c_{1,1}\frac{\epsilon}{b_{1,2}} - c_{0,2}b_{2,1}\frac{\epsilon}{a_{2,2}}
\]
Similarly, 
\[ D_{1,0} = d_{1,0} - c_{2,0}c_{1,1} \epsilon \frac{e}{b_{2,1}} - c_{2,0}b_{1,2} \epsilon \frac{e}{a_{2,2}} \]

We see that \( D_{0,1} \) and \( D_{1,0} \) are \( A \).

For convenience of notation, let \( F = b_{1,1}b_{2,2}b_{1,2}b_{2,1} \), and \( \epsilon = b_{1,2}b_{2,1}a_{2,2} \delta \). Thus \( b_{1,1}b_{2,2} \epsilon = a_{2,2}F \delta \).

\[
\begin{align*}
C_{1,1} &= c_{1,1} + FC_{1,1} \delta \\
D_{1,1} &= d_{1,1} + c_{2,2}F \frac{\delta}{b_{2,2}} \\
D_{0,1} &= d_{0,1} - c_{2,2}F \frac{\delta}{b_{1,2}} \\
D_{1,0} &= d_{1,0} - c_{2,0}F \frac{\delta}{b_{2,1}} \\
D_{0,0} &= d_{0,0} + c_{0,0}F \frac{\delta}{b_{1,1}}
\end{align*}
\]

(2) **Step 2** \( E_{0,0} \) is in \( A \)

**Proof:** Let \( Y = D_{0,0}D_{1,1} - D_{0,1}D_{1,0} \). Thus \( E_{0,0} = \frac{Y}{C_{1,1}} \). Multiply \( Y \) by \( b_{1,1} \), to get

\[
\begin{align*}
b_{1,1}Y &= D_{1,1}(c_{0,0}C_{1,1} - c_{0,1}c_{1,0}) - b_{1,1}D_{0,1}D_{1,0} \\
&= C_{1,1}X_1 - U
\end{align*}
\]

where \( X_1 = D_{1,1}c_{0,0} \), and \( U = c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}D_{1,0} \).

\[
\begin{align*}
b_{2,1}U &= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{2,1}b_{1,1}D_{0,1}D_{1,0} \\
&= b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}(c_{1,0}c_{2,1} - C_{1,1}c_{2,0}) \\
&= c_{1,0}V - C_{1,1}X_2
\end{align*}
\]

where \( X_2 = b_{1,1}D_{0,1}c_{2,0} \) and \( V = b_{2,1}D_{1,1}c_{0,1} + b_{1,1}D_{0,1}c_{2,1} \).
\[ b_{2,2} V = b_{2,2} b_{2,1} D_{1,1} c_{0,1} + b_{2,2} b_{1,1} D_{0,1} c_{2,1} \]
\[ = b_{2,1} c_{0,1} (C_{1,1} c_{2,2} - c_{1,2} c_{2,1}) + b_{2,2} b_{1,1} D_{0,1} c_{2,1} \]
\[ = C_{1,1} X_3 + c_{2,1} W \]

where \( X_3 = b_{2,1} c_{0,1} c_{2,2} \) and \( W = -b_{2,1} c_{0,1} c_{1,2} + b_{2,2} b_{1,1} D_{0,1} \)

\[ b_{1,2} W = -b_{1,2} b_{2,1} c_{0,1} c_{1,2} + b_{2,2} b_{1,1} b_{1,2} D_{0,1} \]
\[ = -b_{1,2} b_{2,1} c_{0,1} c_{1,2} + b_{2,2} b_{1,1} (c_{0,1} c_{1,2} - c_{0,2} C_{1,1}) \]
\[ = -b_{1,1} b_{2,2} c_{0,2} C_{1,1} + c_{0,1} c_{1,2} (-b_{1,2} b_{2,1} + b_{1,1} b_{2,2}) \]
\[ = -b_{1,1} b_{2,2} c_{0,2} C_{1,1} + c_{0,1} c_{1,2} (a_{2,2} C_{1,1} - b_{1,1} b_{2,2} \epsilon) \]
\[ = X_4 C_{1,1} - c_{0,1} c_{1,2} a_{2,2} F \delta \]

where

\[ X_4 = a_{2,2} c_{0,1} c_{1,2} - b_{1,1} b_{2,2} c_{0,2} \]
\[ = a_{2,2} b_{1,2} d_{0,1} - b_{1,2} b_{2,1} c_{0,2} \]
A calculation (reversing the steps above), shows that

\[ FY = \begin{align*}
&b_{1,2}b_{2,2}b_{2,1}C_{1,1}X_1 \\
&- b_{1,2}b_{2,2}C_{1,1}X_2 \\
&- b_{1,2}c_{1,0}C_{1,1}X_3 \\
&- c_{1,0}c_{2,1}C_{1,1}X_4 + c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta
\end{align*} \]

\[ = C_{1,1}(b_{1,2}b_{2,2}b_{2,1}D_{1,1}c_{0,0} \\
- b_{1,2}b_{2,2}b_{1,1}c_{2,0}D_{0,1} \\
- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\
- c_{1,0}c_{2,1}X_4 + c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta)
\]

\[ = C_{1,1}(b_{1,2}b_{2,2}b_{2,1}c_{0,0}(d_{1,1} + c_{2,2} \frac{F}{b_{2,2}}) \\
- b_{1,2}b_{2,2}b_{1,1}c_{2,0}(d_{0,1} - c_{0,2} \frac{F}{b_{1,2}}) \\
- b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2}) \\
+ c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}F\delta
\]
where

\[
p(x) = b_{1,2}b_{2,2}b_{1,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \\
+ c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}
\]

\[
Z = C_{1,1}b_{1,2}b_{2,1}c_{0,0}c_{2,2} \delta \\
- C_{1,1}b_{1,2}b_{2,1}c_{2,0}c_{0,2} \delta \\
+ c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \delta - C_{1,1}a_{2,2}c_{1,1}c_{2,0}c_{0,2} \delta
\]

(1) If we set \( \delta \) to 0, then \( Y \) becomes \( y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0} \), and we have

\[
b_{1,1}b_{2,2}b_{1,2}b_{2,1}y = c_{1,1}p(x)
\]

Each of these expressions \((b_{1,1}b_{2,2}b_{1,2}b_{2,1}, y, c_{1,1}, p(x))\) is a polynomial in the original ring, \( Z[x_{i,j}] \), and \( c_{1,1} \) is indecomposable. Therefore

\[
\frac{p(x)}{F} = \frac{y}{c_{1,1}} = e_{0,0}
\]

(2) The expression for \( Z \) must now be divided by \( C_{1,1} \). The first and second terms are easy:

\[
b_{1,2}b_{2,1}c_{0,0}c_{2,2} \delta = c_{0,0}c_{2,2} \frac{\epsilon}{a_{2,2}}
\]

\[
b_{1,2}b_{2,1}c_{2,0}c_{0,2} \delta = c_{2,0}c_{0,2} \frac{\epsilon}{a_{2,2}}
\]

(3) Let \( P = (c_{1,0}c_{2,1}c_{0,1}c_{2,2} - C_{1,1}c_{1,1}c_{2,0}c_{0,2})a_{2,2} \delta \). Using

\[
c_{1,0}c_{2,1} = b_{2,1}D_{1,0} + c_{2,0}C_{1,1}
\]

\[
c_{0,1}c_{1,2} = b_{1,2}D_{0,1} + c_{0,2}C_{1,1}
\]
we have
\[
P = \left( (b_{2,1}D_{1,0} + c_{2,0} \cdot (b_{1,2}D_{0,1} + c_{0,2}C_{1,1}) - C_{1,1}c_{1,1}c_{2,0}c_{0,2}) \right) a_{2,2}C_{1,1}\delta
\]
\[
= b_{2,1}b_{1,2}D_{1,0}D_{0,1}a_{2,2}\delta + b_{2,1}D_{1,0}c_{0,2}C_{1,1}a_{2,2}\delta + c_{2,0}b_{1,2}D_{0,1}C_{1,1}a_{2,2}\delta + \left( c_{2,0}c_{0,2}C_{1,1} - C_{1,1}c_{1,1}c_{2,0}c_{0,2} \right) a_{2,2}\delta
\]
\[
= D_{1,0}D_{0,1}\epsilon + D_{1,0}c_{0,2}C_{1,1} + C_{1,1}c_{2,0}C_{1,1} - C_{1,1}c_{1,1}c_{2,0}c_{0,2} \cdot \frac{\epsilon}{b_{1,2}}b_{2,1}
\]

(3a) After dividing by \(C_{1,1}\), the first term becomes \(D_{1,0}D_{0,1}\frac{\epsilon}{C_{1,1}}\), which is in \(A\).
(After expanding the definitions of \(D_{1,0}, D_{0,1}\), this will involve some quadratic and cubic terms in \(\epsilon\).)

(3b) After dividing by \(C_{1,1}\), the second term becomes \(D_{1,0}c_{0,2}\frac{\epsilon}{b_{1,2}}\), which is in \(A\).

(3c) After dividing by \(C_{1,1}\), the third term becomes \(c_{2,0}D_{0,1}\frac{\epsilon}{b_{2,1}}\), which is in \(A\).

(3d) After dividing by \(C_{1,1}\), and using \(C_{1,1} - c_{1,1} = \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}}\), the fourth term becomes
\[
c_{2,0}c_{0,2} \cdot \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}} \cdot \frac{\epsilon}{b_{1,2}b_{2,1}}.
\]
Using the identity:
\[
\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1} \frac{\epsilon}{b_{1,2}} + b_{2,1} \frac{\epsilon}{a_{2,2}}
\]
this becomes
\[
c_{2,0}c_{0,2} \cdot \left( c_{1,1} \frac{\epsilon}{b_{1,2}} + b_{2,1} \frac{\epsilon}{a_{2,2}} \right) \frac{\epsilon}{b_{2,1}}
\]
which is in \(A\). (Some terms quadratic in \(\epsilon\) appear.)

***************************************************
To summarize and clarify the situation, this calculation shows that

\[ FY = C_{1,1}p(x) + C_{1,1}FZ_1 + FD_{1,0}D_{0,1} \cdot \epsilon, \]

where

1. \( p(x) \) is the polynomial in \( Z[x_{i,j}] \) that results if \( \epsilon \) is set to 0, and \( \frac{p(x)}{F} = e_{0,0} \).

2. \( Z_1 = \frac{Z}{C_{1,1}} \) is demonstrably in \( R[\frac{\epsilon}{a_{2,2}}, \frac{\epsilon}{b_{1,2}}, \frac{\epsilon}{b_{2,2}}] \).

3. \( \frac{\epsilon}{C_{1,1}} \) is the fraction adjoined to \( A \) at this stage. Therefore,

\[ E_{0,0} = \frac{Y}{C_{1,1}} = e_{0,0} + Z_1 + D_{1,0}D_{0,1} \frac{\epsilon}{C_{1,1}} \text{ is in } A. \]
Step 3 $E_{0,1}$ is in $A$

**Proof:** Let $Y = D_{0,1}D_{1,2} - D_{0,2}D_{1,1}$. We must show that $Y$ is divisible by $C_{1,2}$, and thus $E_{0,1} = \frac{Y}{C_{1,2}}$. Multiply $Y$ by $b_{1,2}$, to get

$$b_{1,2}Y = D_{1,2}(c_{0,1}c_{1,2} - c_{0,2}C_{1,1}) - b_{1,2}d_{0,2}D_{1,1}$$
$$= c_{1,2}X_1 - U$$

where $X_1 = d_{1,2}c_{0,1}$, and $U = c_{0,2}C_{1,1}d_{1,2} + b_{1,2}d_{0,2}D_{1,1}$.

$$b_{2,2}U = b_{2,2}c_{0,2}C_{1,1}d_{1,2} + b_{2,2}b_{1,2}d_{0,2}D_{1,1}$$
$$= b_{2,2}c_{0,2}C_{1,1}d_{1,2} + b_{1,2}d_{0,2}(C_{1,1}c_{2,2} - c_{1,2}c_{2,1})$$
$$= C_{1,1}V - c_{1,2}X_2$$

where $X_2 = b_{1,2}d_{0,2}c_{2,1}$ and $V = b_{2,2}c_{0,2}d_{1,2} + b_{1,2}d_{0,2}c_{2,2}$.

$$b_{2,3}V = b_{2,3}b_{2,2}d_{1,2}c_{0,2} + b_{2,3}b_{1,2}d_{0,2}c_{2,2}$$
$$= b_{2,3}c_{0,2}(c_{1,2}c_{2,3} - c_{1,3}c_{2,2}) + b_{2,3}b_{1,2}d_{0,2}c_{2,2}$$
$$= c_{1,2}X_3 + c_{2,2}W$$

where $X_3 = b_{2,2}c_{0,2}c_{2,3}$ and $W = -b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}d_{0,2}$

$$b_{1,3}W = -b_{1,3}b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}b_{1,3}d_{0,2}$$
$$= -b_{1,3}b_{2,2}c_{0,2}c_{1,3} + b_{2,3}b_{1,2}(c_{0,2}c_{1,3} - c_{0,3}c_{1,2})$$
$$= -b_{1,3}b_{2,3}c_{0,3}c_{1,2} + c_{0,2}c_{1,3}(-b_{1,3}b_{2,2} + b_{1,3}b_{2,3})$$
$$= -b_{1,2}b_{2,3}c_{0,3}c_{1,2} + c_{0,2}c_{1,3}(a_{2,3}c_{1,2})$$
$$= X_4 c_{1,2}$$

where

$$X_4 = a_{2,3}c_{0,2}c_{1,3} - b_{1,2}b_{2,3}c_{0,3}$$
$$= a_{2,3}b_{1,3}d_{0,2} - b_{1,3}b_{2,3}c_{0,3}$$

$$W = (a_{2,3}d_{0,2} - b_{2,2}c_{0,3})c_{1,2} = X_5 c_{1,2}$$

$$b_{2,3}V = c_{1,2}(X_3 + c_{2,2}X_5)$$
$$= c_{1,2}(b_{2,2}c_{0,2}c_{2,3} + c_{2,2}(a_{2,3}d_{0,2} - b_{2,2}c_{0,3}))$$
Proof: By a shift of indices, it is sufficient to prove the assertion when $i, j = 0, 0$. That is, let

\[
b_{i+1,j+1}c_{i,j} = b_{i,j}b_{i+1,j+1} - b_{i,j+1}b_{i+1,j}
\]
\[
b_{i+1,j+1}d_{i,j} = c_{i,j}c_{i+1,j+1} - c_{i,j+1}c_{i+1,j}
\]

Then $d_{i,j} \cdot d_{i+1,j+1} - d_{i,j+1} \cdot d_{i+1,j}$ is divisible by $c_{i+1,j+1}$ in $R$.

**Proof:** By a shift of indices, it is sufficient to prove the assertion when $i, j = 0, 0$. That is, let $Y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0}$. We must show that $Y$ is divisible by $c_{1,1}$, and that the quotient is in $R$. Multiply $Y$ by $b_{1,1}$, to get

\[
b_{1,1}Y = d_{1,1}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}) - b_{1,1}d_{0,1}d_{1,0}
\]
\[= c_{1,1}X_1 - U
\]

where $X_1 = d_{1,1}c_{0,0}$, and $U = c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}d_{1,0}$.

\[
b_{2,1}U = b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{2,1}b_{1,1}d_{0,1}d_{1,0}
\]
\[= b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}(c_{1,0}c_{2,1} - c_{1,1}c_{2,0})
\]
\[= c_{1,0}V - c_{1,1}X_2
\]

where $X_2 = b_{1,1}d_{0,1}c_{2,0}$ and $V = b_{2,1}d_{1,1}c_{0,1} + b_{1,1}d_{0,1}c_{2,1}$.
\[ b_{2,2}V = b_{2,2}b_{2,1}d_{1,1}c_{0,1} + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \]
\[ = b_{2,1}c_{0,1}(c_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + b_{2,2}b_{1,1}d_{0,1}c_{2,1} \]
\[ = c_{1,1}X_3 + c_{2,1}W \]

where \( X_3 = b_{2,1}c_{0,1}c_{2,2} \) and \( W = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}d_{0,1} \)

\[ b_{1,2}W = -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}(c_{0,1}c_{1,2} - c_{0,2}c_{1,1}) \]
\[ = -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2}) \]
\[ = -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(a_{2,2}c_{1,1}) \]
\[ = (a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2})c_{1,1} \]
\[ = (a_{2,2}d_{0,1} - b_{1,2}c_{0,2})b_{1,2}c_{1,1} \]

Thus \( W \) is divisible by \( c_{1,1} \), then \( V \), then \( U \), and finally \( Y \) is divisible by \( c_{1,1} \).

The quotient \( \frac{Y}{c_{1,1}} \) will be called \( e_{0,0} \). By a shift of indices, \( e_{i,j} \) may be defined in \( R \) for all \( i, j \). In this way, the planes are continued indefinitely \( E, F, G, \ldots \) Similarly, the planes can be continued in the other direction \( B, A, Z, Y, \ldots \)