1 **1 Hankel Determinants**

A sequence of indeterminants $x_0, x_1, \ldots$ is used to make a a Hankel array:

$$\begin{array}{llll} x_0 & x_1 & x_2 & x_3 \ldots \\ x_1 & x_2 & x_3 & x_4 \ldots \\ x_2 & x_3 & x_4 & x_5 \ldots \\ x_3 & x_4 & x_5 & x_6 \ldots \end{array}$$

For each $i \geq 0$, and $n \geq i$, $H_{n,i}$ is defined to be the $i \times i$ connected minor with x_n along the back diagonal.

$$h_{n,i} = \det H_{n,i}$$

The c-table is the set of $h_{n,i}$ for $n + 1 \geq i$. these determinants satisfy the Sylvester identity:

$$h_{n,i-1} \cdot h_{n,i+1} = h_{n-1,i} \cdot h_{n+1,i} - h_{n,i}^2$$

2 A pattern of locations in the c-table of the form

$$\begin{array}{ccc} & N & \\ W & C & E \\ & S & \end{array}$$

is called a Sylvestor diamond.

$$n \cdot s = w \cdot e - c^2$$

Suppose the following constellation occurs in the c-table, C is located as (s, i).

$$\begin{array}{ccccc} & & NN & & \\ & NW & N & NE & \\ WW & W & C & E & EE \\ & SW & S & SE & E \\ & & SS & & \end{array}$$

Let Λ be the 3×3 matrix whose entries are at the positions:

$$\begin{bmatrix} NN & NE & EE \\ NW & C & SE \\ WW & SW & SS \end{bmatrix}$$

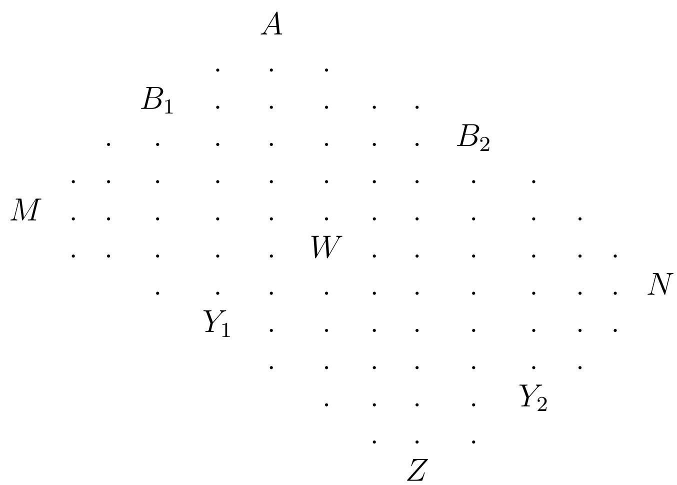
3 Gragg's identity is

$$c \cdot \det(\Lambda) = -c^2(w \cdot e + n \cdot s)$$

This identity is derived solely from Sylvester's identity (5 times). If $x_0, x_1, \ldots$ are indeterminates in the ring $\mathbf{Z}[x_0, x_1, \ldots]$, then each Hankel det is an irreducible poly in the $x_0, x_1, \ldots$. One c may be cancelled to give:

$$\det(\Lambda) = -c(w \cdot e + n \cdot s)$$

Consider a larger region in the c-table. Suppose $A = (s, i)$ and $Z = (t, i')$ are locations as in the diagram:



4 Thus A is at the location of the $i \times i$ matrix with x_s on the back diagonal , and Z is at the location of the $i' \times i'$ matrix with x_t on the back diagonal.

Let Λ be the matrix formed by putting the SW diagonal from A as the left hand column and the SE diagonal as the top row:

$$\Lambda = \begin{bmatrix} a & . & . & . & b_2 & . & . & . & n \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ b_1 & . & . & . & w & . & . & . & y_2 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ m & . & . & . & y_1 & . & . & . & z \end{bmatrix}$$

This matrix Λ is indexed so that $\Lambda[1, 1] = a$ and $\Lambda[j, k] = z$. The notation $\Lambda[g, h, i; l, m, n]$ will stand for the submatrix of Λ withe columns g, h, i and rows l, m, n For each pair of locations A and Z as above, there is a polynomial $\phi_{A,Z}$ in $\mathbf{Z}[x_0, x_1, \ldots]$. These polynomials have the following property. Let j' be a row index with $1 < j' < j$, and k' be a columns index with $1 < k' < k$.

For each pair of locations A and Z, there is a poly $\phi_{A,Z}$ with the following property: Let j' be a row index with $1 < j' < j$ let k' be a column index with $1 < k' < k$. Refer to the diagram for the locations of B_1, B_2, Y_1, Y_2, W. Thus

$$\Lambda[1, j', j; 1, k', k] = \begin{bmatrix} a & b_2 & n \\ b_1 & w & y_2 \\ m & y_1 & z \end{bmatrix}$$

5 **Theorem** There are two expressions for the det of this matrix:

$$\begin{array}{lrcl} (i) & \det \Lambda[1, j', j; 1, k', k] & = & y_1 \phi_{A,Y_2} + y_2 \phi_{A,Y_1} + z \phi_{A,W} - w \phi_{A,Z} \\ (ii) & \det \Lambda[1, j', j; 1, k', k] & = & b_1 \phi_{B_2,Z} + b_2 \phi_{B_1,Z} + a \phi_{W,Z} - w \phi_{A,Z} \end{array}$$

We need to consider in more detail the region which has A and Z at opp corners of a rectangle. From Λ, 4 rows are seleted, containing A; B_1 (and C); Y_2 and V; and Z. Four columns are selected containing A; B_2 (and C; Y_1 (and V; Z.

$$\begin{array}{cccccccccccc}
 & & & & & A & & & & & & \\
 & & & & . & . & . & & & & & \\
 & & & B_1 & . & . & . & B_2 & & & & \\
 & & . & . & . & . & . & . & . & & & \\
 & . & . & . & . & C & . & . & . & . & & \\
M & . & . & . & . & . & . & . & . & . & . & \\
 & . & . & . & . & . & . & . & . & . & . & N \\
 & & . & . & . & . & V & . & . & . & . & \\
 & & & . & . & . & . & . & . & . & & \\
 & & & & Y_1 & . & . & . & Y_2 & & & \\
 & & & & & . & . & . & & & & \\
 & & & & & & Z & & & & &
\end{array}$$

6 Let Λ bethe matrix by putting the SW diagonal emanating from A as the LH column and the SE diagonal as the top row.

$$\Lambda = \begin{bmatrix}
a & . & b_2 & . & \cdots & . & . & . & n \\
. & . & . & . & \cdots & . & . & . & . \\
b_1 & . & c & . & \cdots & . & . & . & . \\
. & . & . & . & \cdots & . & . & . & . \\
. & . & . & . & \cdots & . & . & . & . \\
. & . & . & . & \cdots & . & v & . & y_2 \\
. & . & . & . & \cdots & . & . & . & . \\
m & . & . & . & \cdots & . & y_2 & . & z
\end{bmatrix}$$

These four rows and columns define a matrix, which we call Γ. Thus

$$\Gamma = \begin{bmatrix} a & b_2 & . & . \\ b_1 & c & . & . \\ . & . & v & y_2 \\ . & . & y_1 & z \end{bmatrix}$$

7 Polynomials $\Omega_{A,V,Z}$ and $\Upsilon_{Z,C,A}$ will be defined inductively depending on the locations A, C, V, and Z. These poly's and $\phi_{A,Z}$ and $\phi_{Z,A}$ are defined only when $t-s+j-1$ is an even integer and $j-i \geq 1$.

(i) If Z is on either SW or SE diagonal, then all four are 0

(ii) Suppose $A=(s,i)$, $C=V=(s,i+2)$ and $Z=(z,i+4)$. Then let p be the poly

$$p = -(h_{s-1,i+2} \cdot h_{s+1,i+2} + (h_{s1,i+1} \cdot h_{s,i+3})$$

These are the Hankel dets to W, E, N, and S of position C. Then

$$\Omega_{A,V,Z} = v \cdot p = c \cdot p = \Upsilon_{Z,C,A}$$

$$\phi_{S,T} = p = \phi_{T,S}$$

(iii) Suppose inductively that $\phi_{S,T}$ and $\phi_{T,S}$ have been defined when the difference between S and T is strictly less than the difference between A and Z. Then $\Omega_{A,V,Z}$ and $\Upsilon_{Z,C,A}$ are defined by

$$\begin{aligned} \Omega_{A,V,Z} &= y_1\phi_{A,Y_2} + y_2\phi_{A,Y_1} - z\phi_{A,V} - \Delta[1,3,4;1,3,4] \\ \Upsilon_{Z,C,A} &= b_1\phi_{Z,B_2} + b_2\phi_{Z,B_1} - z\phi_{Z,C} - \Delta[1,2,4;1,2,4] \end{aligned}$$

8 (iv) The expressions $\phi_{A,Z}$ and $\phi_{Z,A}$ are defined by

$$\begin{aligned} \phi_{A,Z} &= \frac{\Omega_{A,V,Z}}{v} \\ \phi_{Z,A} &= \frac{\Omega_{Z,C,A}}{c} \end{aligned}$$

Lemma 1 There is a polynomial identity

$$\Omega_{A,V,Z} \cdot c = \Upsilon_{Z,C,A} \cdot v$$

8 • Both c and v are Hankel dets and hence are irreducible polys in the x_i. The identity of Lemma 1 implies that if $V \neq C$, there is an equality

$$\phi_{A,Z} = \frac{\Omega_{A,V,Z}}{v} = \frac{\Upsilon_{Z,C,A}}{c} = \phi_{Z,A}$$

The case $V = C$ is handled by (ii) above. In this case Gragg's identity implies that

$$\Omega_{A,V,Z} = -\Delta[1,2,3;1,2,3] = \Upsilon_{Z,C,A}$$

9 The proof of Lemma 1 makes use of the 8-term identity (below). Recall that

$$\Gamma = \begin{bmatrix} a & b_2 & . & . \\ b_1 & c & . & . \\ . & . & v & y_2 \\ . & . & y_1 & z \end{bmatrix}$$

The determinant of $\Gamma[g, h, i; l, m, n]$ will be denoted $\Delta[g, h, i; l, m, n]$. The proof uses the 8-term identity among the 3 by 3 dets of Γ.

Lemma 2 There is an 8 term identity:

$$\begin{aligned} \Delta[2,3,4;2,3,4]\cdot a - \Delta[2,3,4;1,3,4]\cdot b_2 & \\ -\Delta[1,3,4;2,3,4]\cdot b_1 + \Delta[1,3,4;1,3,4]\cdot c & \\ -\Delta[1,2,4;1,2,4]\cdot w + \Delta[1,2,4;1,2,3]\cdot y_2 & \\ +\Delta[1,2,3;1,2,4]\cdot y_1 - \Delta[1,2,3;1,2,3]\cdot z &= 0 \end{aligned}$$

The proof of 8-term id uses the cofactor expansion by the first and second rows, and by the third and fourth columns. After taking the sum, and cancelling common terms the id follows. Observe that the uncancelled terms are the cofactors of 8 entries of the matrix.

Proof of Lemma 1 If Z is on either the SW or the SE diagonal from A, then

$$\phi_{A,Z}, \Omega_{A,V,Z}, \Upsilon_{Z,C,A}, \phi_{Z,A}$$

are all defined to be 0. For Z strictly within the SW and SE diagonals, teh proof is by induction on diff levels A and Z. Start at diff $= 4$. Thus $A = (s, i)$ and $Z = (s, i+4)$. Then $C = V = (s, i+2), c = v$ In this case $\phi_{A,Z}, \Omega_{A,V,Z}, \Upsilon_{Z,C,A}, \phi_{Z,A}$ defined by (ii) and statement is Gragg's id.

Suppose diff A and Z is k, with $k > 4$. Then

$$\begin{aligned}
\Omega_{A,V,Z} \cdot c &= y_1\phi_{A,y_2} \cdot c + y_2\phi_{A,Y_1} \cdot c \\
&\quad -z\phi_{A,V} \cdot c - \Delta[1,3,4;1,3,4] \cdot c \\
&= y_1\{b_1\phi_{Y_2,B_2} + b_2\phi_{Y_2,B_1} \\
&\quad -a\phi_{Y_2,C} - \Delta[1,2,3;1,2,4]\} \\
&\quad +y_2\{b_1\phi_{Y_2,C} - \Delta[1,2,3;1,2,4] \\
&\quad -a\phi_{Y_1,C} - \Delta[1,2,4;1,2,3]\} \\
&\quad -z\{b_1\phi_{V,B_2} + b_2\phi_{V,B} \\
&\quad -a\phi_{V,C} - \Delta[1,2,3;1,2,3]\} \\
&\quad -\Delta[1,3,4;1,3,4] \cdot c
\end{aligned}$$

$$\begin{aligned}
\Upsilon_{Z,C,A} \cdot v &= b_1\phi_{Z,B_2} \cdot v + b_2\phi_{Z,B_1} \cdot v \\
&\quad -a\phi_{Z,C} \cdot v - \Delta[1,2,4;1,2,4] \cdot v \\
&= b_1\{y_1\phi_{B_2,Y_2} + y_2\phi_{B_2,Y_1} \\
&\quad -z\phi_{B_2,V} - \Delta[2,3,4;1,3,4]\} \\
&\quad +b_2\{y_1\phi_{B_1,Y_2} + y_2\phi_{B_1,Y_1} \\
&\quad -z\phi_{B_1,V} - \Delta[1,3,4;2,3,4]\} \\
&\quad -a\{y_1\phi_{C,Y_2} + y_2\phi_{C,Y_1} \\
&\quad -z\phi_{C,V} - \Delta[2,3,4;2,3,4]\} \\
&\quad -\Delta[1,2,4;1,2,4] \cdot v
\end{aligned}$$

The equality $\Omega_{A,V,Z} \cdot c = \Upsilon_{Z,C,A} \cdot v$ follows from the 8-term id, and repeated use (9 times) of $\phi_{S,T} = \phi_{T,S}$. It follows that

$$\frac{\Omega_{A,V,Z}}{v} = \frac{\Upsilon_{Z,C,A}}{c}$$

11 The LHS depends only on the choice of rows and columns $1,3,4$ of Γ, which are the rows $1, j", j$ and columns $1, k", k$ of Λ. Similarly, The RHS depends only on the choice of rows and columns $1,2,4$ of Γ, which are the rows $1, j', j$ and columns $1, k', k$ of Λ. Consequently, the poly $\frac{\Omega_{A,V,Z}}{v}$ is the same for all choices of row $j"$ and column

k" of Λ. Similarly, the fraction $\dfrac{\Upsilon_{Z,C,A}}{c}$ is the same for all choices row j' and column k' of Λ. This common quotient is the definition of $\phi_{A,Z} = \phi_{Z,A}$.
A special case occurs when $A = (s,i)$ $C = (s,i+2) = V$, and $Z = (s,i+4)$. Then equation - becomes

$$\det[1,2,3;1,2,3] = c \cdot \Phi_{A,Z}$$

where $\phi_{A,Z} = -(we+ns)$

- It follows from Lemma 1, that there is a polynomial identity

$$\Delta[1,3,4;1,3,4] = v\phi_{A,Z} - y_1\phi_{A,Y_2} - y_2\phi_{A,Y_1} + z\phi_{A,V}$$

12 **Example**

Suppose a portion of the table is

$$\Lambda = \begin{bmatrix} a_n & b_{n+1} & c_{n+2} & d_{n+3} & . & . \\ b_{n-1} & c_n & d_{n+1} & e_{n+2} & . & . \\ c_{n-2} & d_{n-1} & e_n & f_{n+1} & . & \\ d_{n-3} & e_{n-2} & f_{n-1} & g_n & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

A special case of the 8-term identity is

$$\begin{aligned} \Delta[2,3,4;1,2,3] \cdot b_{n+1} - \Delta[1,3,4;1,2,3] \cdot c_n \quad & + \\ \Delta[1,2,4;1,2,3] \cdot d_{n-1} \quad & - \quad \Delta[1,2,3;1,2,3] \cdot e_{n-2} = 0 \end{aligned}$$

This can also be seen by obs that it is the exp of a 4 by 4 det with two equal columns. Gragg's id centered at C_n and D_{n-1} are

$$\begin{aligned} \det \Delta[1,2,3;1,2,3] &= -c_n \cdot r \\ \det \Delta[2,3,4;1,2,3] &= -d_{n-1} \cdot s \end{aligned}$$

where r and s are each of the form (we - ns) for appropriate Sylvester Diamonds centered at C_n and D_{n-1} respectively. Using these identities, and rearranging terms, we have

$$(r \cdot e_{n-2} - \Delta[1,3,4;1,2,3])\, c_n = (s \cdot b_{n+1} - \Delta[1,2,4;1,2,3])\, d_{n-1}$$

In the notation of Lemma 1, $r = \phi_{A,E_n}$ and $s = \phi_{F_{n-1},B_{n-1}}$, and this equation would be

$$\Omega_{A,D_{n-1},F_{n-1}} \cdot c_n = \Upsilon_{F_{n-1},C_n,A} \cdot d_{n-1}$$

Let p_1 be the GCD of $\Omega_{A,D_{n-1},F_{n-1}}$ and $\Upsilon_{F_{n-1},C_n,A}$ Then

$$p_1 = \phi_{A,F_{n-1}} = \frac{\Omega_{A,D_{n-1},F_{n-1}}}{d_{n-1}} = \frac{\Upsilon_{F_{n-1},C_n,A}}{c_n} = \phi_{F_{n-1},A}$$

and

$$p_1 \cdot d_{n-1} = e_{n-2} \cdot s - \Delta[1,3,4;1,2,3]$$

There is a similar expression

$$e_{n+2} \cdot s' - \Delta[1,2,3;1,3,4] = p_2 \cdot d_{n+1}$$

For the location G_n we have

$$\Omega_{A_n,E_n,G_n} = g \cdot r - f_{n-1} \cdot p_2 - f_{n+1} \cdot p_1 - \Delta[1,3,4;1,3,4]$$

Reversing the roles of A_n and G_n there are polynomials r', q_1, and q_2 with

$$\Upsilon_{G_n,C_n,A_n} = a \cdot r' - b_{n-1} \cdot q_2 - b_{n+1} \cdot q_1 - \Delta[1,2,4;1,2,4]$$

A calculation similar to that in Lemma 1, shows that

$$\Omega_{A_n,E_n,G_n} \cdot c_n = \Upsilon_{G_n,C_n,A_n} \cdot e_n$$

In the notation of – and –, ϕ_{A_n,G_n} is the GCD of the LHS and the RHS. Thus

$$\phi_{A_n,G_n} = \frac{\Omega_{A_n,C_n,G_n}}{e_n} = \frac{\Upsilon_{G_n,E_n,A_n}}{c_n}$$

13 Suppose ϵ is the error at C_n. Then the error at G_n will be

$$\delta(g_n) = \left[\frac{\phi_{A_n,G_n} + a_n g_n - d_{n-3} d_{n+3}}{b_{n-1} b_{n+1}} + \frac{a}{c}\left(\frac{e_n^2 - e_{n-2} e_{n+2}}{b_{n-1} b_{n+1}}\right)\right] \cdot \epsilon$$

• For any position X, let $\delta(x)$ be the change at X that results from a change of $\delta(c)$ at C. Assume the calculations are made in char 2, and that second order effects can be ignored.With these assumptions,

$$\delta(z) = \left[\frac{\phi_{A,Z} - az + mn}{b_1 b_2} + \frac{a}{c}\left(\frac{\phi_{C,Z} + m_2 n_2}{b_1 b_2}\right)\right] \cdot \delta(c)$$

14 **The FPA** A fixed precision of p bits is chosen. A FPE is (m, e), where

(i) m is an odd integer of p bits;

(ii) e is an integer exponent.

Each entry in the c-table is a FPE, computed by the FPA. Specifically, s is computed as $S = \dfrac{w \cdot e - c^2}{n}$

16 **Robbins Conjecture** Suppose the c-table is computed by the FPA with mantissa length p, and suppose the largest exponent in the computed c-table is q. Then each computed expression is accurate in the lowest $p - q$ bits.

The expression (m, e) corresponds to the polynomial

$$(m_{p-1}t^{p-1} + \ldots + m_2 t^2 + m_1 t + 1) \cdot t^e$$

In every case the mantissa has p bits, with the lowest bit always 1.

$$\begin{array}{ccc} & N & \\ W & C & E \\ & S & \end{array}$$

$$s = \frac{w \cdot e - c^2}{n}$$

Neglecting second order effects,

$$\begin{aligned} \delta(s) &= \frac{e \cdot \delta(w) + w \cdot \delta(e) - 2 \cdot \delta(c) \cdot c}{n} - \frac{(we - c^2) \cdot \delta(n)}{n^2} \\ &= \frac{e \cdot \delta(w) + w \cdot \delta(e) - s \cdot \delta(n)}{n} \end{aligned}$$

17 A portion of the c-table.

$$
\begin{array}{ccccccccccc}
 & & & & & A & & & & & \\
 & & & & B_{n-1} & . & B_{n+1} & & & & \\
 & & & C_{n-2} & . & C_n & . & C_{n+2} & & & \\
 & & D_{n-3} & . & D_{n-1} & . & D_{n+1} & . & D_{n+3} & & \\
 & E_{n-4} & . & E_{n-2} & . & E_n & . & E_{n+2} & . & E_{n+4} & \\
. & . & F_{n-3} & . & F_{n-1} & . & F_{n+1} & . & F_{n+3} & . & . \\
. & . & . & . & . & G_n & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & .
\end{array}
$$

Here the dots are locations at the centers of the Sylvester diamonds.

18 **Proposition** (p 21) Suppose that at location $C_n = (n, i)$, c_n is changed to $c_n + \epsilon$. Assume that above C_n the computations are exact, and that the lower levels are computed with high precision. Assume also that second order effects can be neglected. Let

$$
r = \max \begin{cases} \operatorname{ord}(b_{n-1}) + \operatorname{ord}(b_{n+1}) \\ \operatorname{ord}(b_{n-1}) + \operatorname{ord}(b_{n+1}) + \operatorname{ord}(c_n) - \operatorname{ord}(a_n) \end{cases} \tag{1}
$$

Then the change at any position in the c-table has order at least

$$
\operatorname{ord}(\epsilon) - r
$$

Change at D_n SI is used in the computation

$$
\begin{aligned}
b_n D_n &= c_{n-1}c_{n+1} - C_n^2 \\
&= c_{n-1}c_{n+1} - (c_n + \epsilon)^2 \\
&= c_{n-1}c_{n+1} - c_n^2 - 2c_n\epsilon - \epsilon^2 \\
&= b_n d_n - 2c_n\epsilon - \epsilon^2
\end{aligned}
$$

$$
D_n = d_n - \frac{2c_n \cdot \epsilon + \epsilon^2}{b_n}
$$

Change at D_{n-1} SI is used in the computation

$$
b_{n-1}d_{n-1} = c_{n-2}c_n - c_{n-1}^2
$$

$$
\begin{aligned}
b_{n-1}D_{n-1} &= c_{n-2}C_n - c_{n-1}^2 \\
&= c_{n-2}(c_n + \epsilon) - c_{n-1}^2 \\
&= c_{n-2}c_n + c_{n-2}\epsilon - c_{n-1}^2 \\
&= b_{n-1}d_{n-1} + c_{n-2}\epsilon
\end{aligned}
$$

$$
D_{n-1} = d_{n-1} + \frac{c_{n-2}\epsilon}{b_{n-1}}
$$

Changes along diagonals

The SW diagonal is $\{C_n, D_{n-1}, E_{n-2}, F_{n-3}, \ldots\}$
The SW diagonal is $\{C_n, D_{n+1}, E_{n+2}, F_{n+3}, \ldots\}$

The computation of e_{n-2} is

$$
c_{n-2}e_{n-2} = d_{n-3}d_{n-1} + d_{n-2}^2
$$

Then

$$
\Delta(e_{n-2}) = \frac{d_{n-3}}{c_{n-2}} \cdot \Delta d_{n-1} = \left(\frac{d_{n-3}}{c_{n-2}} \cdot \frac{c_{n-2}}{b_{n-1}}\right) \cdot \epsilon = \left(\frac{d_{n-3}}{b_{n-1}}\right) \cdot \epsilon
$$

$$
\Delta(f_{n-3}) = \frac{e_{n-4}}{d_{n-3}} \cdot \Delta e_{n-2} = \left(\frac{e_{n-4}}{d_{n-3}} \cdot \frac{d_{n-3}}{b_{n-1}}\right) \cdot \epsilon = \frac{e_{n-4}}{b_{n-1}} \cdot \epsilon
$$

For each position on the diagonal emansating from C_n,

$$
\begin{aligned}
\Delta(h_{n-k,i+k}) &= \frac{h_{n-k-1,i+k+1}}{b_{n+1}} \cdot \epsilon \\
\Delta(h_{n+k,i+k}) &= \frac{h_{n+k-1,i+k+1}}{b_{n+1}} \cdot \epsilon
\end{aligned}
$$

Change at E_n. E_n is defined by $C_nE_n = D_{n-1}D_{n+1} + D_n^2$. Let $Y = D_{n-1}D_{n+1} - D_n^2$
We must show Y is divisible by c_n.

$$
\begin{aligned}
b_{n-1}Y &= (c_{n-2}C_n - c_{n-1}^2)D_{n+1} - b_{n-1}D_n^2 \\
&= c_{n-2}c_nD_{n+1} - c_{n-1}^2D_{n+1} - b_{n-1}D_n^2 \\
&= X_1C_n - U
\end{aligned}
$$

where $X_1 = c_{n-2}D_{n+1}$ and $U = c_{n-1}^2D_{n+1} + b_{n-1}D_n^2$

$$
\begin{aligned}
b_{n+1}U &= c_{n-1}^2(C_n c_{n+2} - c_{n+1}^2) + b_{n+1}b_{n-1}D_n^2 \\
&= c_{n-1}^2 c_{n+2} C_n - c_{n-1}^2 c_{n+1}^2 + b_{n-1}b_{n+1}D_n^2 \\
&= X_2 C_n + V
\end{aligned}
$$

where $X_2 = c_{n-1}^2 c_{n+2}$ and $V - c_{n-1}^2 c_{n+1}^2 + b_{n-1}b_{n+1}D_n^2$

$$
\begin{aligned}
V &= -c_{n-1}^2 c_{n+1}^2 + b_{n-1}b_{n+1}D_n^2 \\
&= -(b_n D_n + c_n^2)^2 + b_{n-1}b_{n+1}D_n^2 \\
&= -b_n^2 D_n^2 - 2b_n c_n D_n^2 - c_n^4 + b_{n-1}b_{n+1}D_n^2 \\
&= C_n X_3 + (a_n C_n + b_{n-1}b_{n+1}\epsilon)D_n^2 \\
&= C_n X_4 + b_{n-1}b_{n+1}D_n^2\epsilon
\end{aligned}
$$

where $X_4 = X_3 + a_n D_n^2$

Change at E_{n-1} The computation of e_{n-1} is

$$c_{n-1}e_{n-1} = D_{n-2}D_n + D_{n-1}^2$$

Let $Y = D_{n-2}D_n - D_{n-1}^2$ We must show Y is divisible by c_{n-1}.

$$
\begin{aligned}
b_{n-2}Y &= (c_{n-3}C_{n-1} - c_{n-2}^2)D_n - b_{n-2}D_{n-1}^2 \\
&= c_{n-3}c_{n-1}D_n - c_{n-2}^2 D_n - b_{n-2}D_{n-1}^2 \\
&= X_1 c_{n-1} - U
\end{aligned}
$$

where $X_1 = c_{n-3}D_n$ and $U = c_{n-2}^2 D_n + b_{n-2}D_{n-1}^2$

$$
\begin{aligned}
b_n U &= c_{n-2}^2(C_{n-1}c_{n+1} - C_n^2) + b_{n+2}b_n D_{n-1}^2 \\
&= c_{n-2}^2 c_{n+1}C_{n-1} - c_{n-2}^2 C_n^2 + b_{n-2}b_n D_{n-1}^2 \\
&= X_2 c_{n-1} + V
\end{aligned}
$$

where $X_2 = c_{n-2}^2 c_{n+1}$ and $V - c_{n-2}^2 c_n^2 + b_{n-2}b_n D_n^2$

$$
\begin{aligned}
V &= -c_{n-2}^2 C_n^2 + b_{n-2}b_n D_{n-1}^2 \\
&= -(b_{n-1}D_{n-1} + c_{n-1}^2)^2 + b_{n-2}b_n D_{n-1}^2 \\
&= -b_{n-1}^2 D_{n-1}^2 - 2b_{n-1}c_{n-1}D_{n-1}^2 - c_{n-1}^4 + b_{n-2}b_n D_{n-1}^2 \\
&= c_{n-1}X_3 + a_{n-1}c_{n-1}D_{n-1}^2 \\
&= c_{n-1}X_4
\end{aligned}
$$

where $X_4 = X_3 + a_n D_n^2$

E_{n+1} is similar to E_{n-1}.

The first interesting case occurs at E_n. This is a special case of the inductive step (later). It must be dealt with separately, because it is the starting point of an induction based on the diffeence between A and Z. The value of E_n is

$$c_n e_n = d_{n-1}d_{n-1} - d_n^2$$

$$
\begin{aligned}
\Delta(e_n) &= \frac{d_{n-1}\Delta(d_{n+1})}{c_n} + \frac{d_{n+1}\Delta(d_{n-1})}{c_n} - \frac{e_n\Delta(c_n)}{c_n} \\
&= \left(\frac{d_{n-1}c_{n+2}b_{n-1} + d_{n+1}c_{n-2}b_{n+1} - e_n b_{n-1}b_{n+1}}{c_n}\right)\cdot\epsilon
\end{aligned}
$$

Let M be the matrix:

$$M = \begin{bmatrix} a_n & b_{n+1} & c_{n+2} \\ b_{n-1} & c_n & d_{n+1} \\ c_{n-2} & d_{n-1} & e_n \end{bmatrix}$$

Gragg's identity centered at C_n is $\det(M) = p\cdot c_n$, where $p = -(ns+ew)$, and n, s, w, e are the entries neighboring c_n. Thus the change at E_n can be rewritten as

$$
\begin{aligned}
\Delta(e_n) &= \frac{pc_n - a_n c_n e_n + c_{n+2}c_n c_{n-2} + a_n d_{n-1}d_{n+1}}{c_n b_{n-1}b_{n+1}} \\
&= \left(\frac{p - a_n e_n + c_{n+2}c_{n-2}}{b_{n-1}b_{n+1}}\right)\cdot\epsilon + \left(\frac{a_n d_{n-1}d_{n+1}}{c_n b_{n-1}b_{n+1}}\right)\cdot\epsilon
\end{aligned}
$$

The first fraction has order $\geq \operatorname{ord}(\epsilon) - \operatorname{ord}(b_{n-1}b_{n+1})$

The second fraction has order $\geq \operatorname{ord}(\epsilon + \operatorname{ord}(a_n)) - \operatorname{ord}(c_n) - \operatorname{ord}(b_{n-1}b_{n+1})$

Taken together, thses show that

$$\operatorname{ord}(\Delta(e_n)) \geq \operatorname{ord}(\epsilon) - r$$

25 **Change at Z**

$$\Lambda = \begin{bmatrix} a & b_2 & . & . & \cdots & . & . & n_1 & n \\ b_1 & c & . & . & \cdots & . & . & n_3 & n_2 \\ . & . & e & . & \cdots & . & . & . & . \\ . & . & . & . & \cdots & . & . & . & . \\ . & . & . & . & \cdots & . & . & . & . \\ . & . & . & . & \cdots & . & . & . & . \\ . & . & . & . & \cdots & . & . & . & . \\ m_1 & m_3 & . & . & \cdots & . & . & v & y_2 \\ m & m_2 & . & . & \cdots & . & . & y_1 & z \end{bmatrix}$$

We want to show that

$$\Delta(z) = \left[\frac{\phi_{A,Z} - az + mn}{b_1 b_2} + \frac{a}{c}\left(\frac{\phi_{C,Z} + m_2 n_2}{b_1 b_2}\right)\right] \cdot \epsilon$$

The expression for $\Delta(z)$ can be simplified as follows. For any two locations$S$ and T let M and N be the locations at the opposite corners of the rectangle determined by S and T. let

$$\psi_{S,T} = \phi_{S,T} + mn$$

With this substitution, the identity becomes

$$y_1 \psi_{A,Y_2} + y_2 \psi_{A,Y_1} - z\psi_{A,V} - v\psi_{A,Z} = -avz + ay_1 y_2$$

The expression for Δz becomes

$$\Delta(z) = \left(\frac{c\psi_{A,Z} - acz + a\psi_{C.Z}}{cb_1 b_2}\right) \cdot \epsilon$$

When Z is at location E $\psi_{C,E} = -d_{n-1}d_{n+1}$ It was established that

$$\begin{aligned}\Delta(e_n) &= \left(\frac{c_n^3 + a_n c_n e_n - c_{n+2}c_n c_{n-2} - a_n d_{n-1}d_{n+1}}{c_n b_{n-1} b_{n+1}}\right)\cdot \epsilon \\ &= \left(\frac{c_n \psi_{A_n,E_n} + a_n c_n e_n + a_n \psi_{C_n,E_n}}{c_n b_{n-1} b_{n+1}}\right)\cdot \epsilon\end{aligned}$$

Start of induction based on diff between A and Z. The value of z is computed as

$$z = \frac{y_1 \cdot y_2 - y_0^2}{v}$$

26 In Char 2, the change at C causes no change at Y_0 (to the first order effects). Inductively, assume that

$$\Delta(y_1) = \left(\frac{c\psi_{A,Y_1} - acy_1 + a\psi_{C,Y_1}}{cb_1b_2}\right)\cdot \epsilon$$

$$\Delta(y_2) = \left(\frac{c\psi_{A,Y_2} - acy_2 + a\psi_{C,Y_2}}{cb_1b_2}\right)\cdot \epsilon$$

$$\Delta(v) = \left(\frac{c\psi_{A,V} - acv + a\psi_{C,V}}{cb_1b_2}\right)\cdot \epsilon$$

$$\begin{aligned}\Delta(z) &= \left(\frac{y_1\delta(y_2) + y_2\delta(y_1) - z\Delta(v)}{v}\right)\cdot \epsilon + \\ &\quad \left(\frac{cy_1\psi_{A,Y_2} - y_1 acy_2 + ay_1\psi_{C,Y_2}}{vcb_1b_2}\right)\cdot \epsilon + \\ &\quad \left(\frac{cy_2\psi_{A,Y_1} - y_2 acy_1 + ay_2\psi_{C,Y_1}}{vcb_1b_2}\right)\cdot \epsilon + \\ &\quad \left(\frac{-cz\psi_{A,V} - zacv - za\psi_{C,V}}{vcb_1b_2}\right)\cdot \epsilon \\ &= \left(\frac{c\psi_{A,Z} - acz + a\psi_{C,Z}}{cb_1b_2}\right)\cdot \epsilon\end{aligned}$$

SI is used in

$$c = \frac{b_1 b_2 - b_0^2}{a}$$

where $b_1 b_2 - b_0^2$ is computed first, and then the result is divided by a. An error can originate at C only if ord $\quad b_1 b_2 = \mathrm{ord} b_0^2$ Suppose this occurs. Each mantissa is an odd integer with p bits, where $p =$ precision. Therefore

$$\mathrm{ord}\Delta(c) \geq p + \mathrm{ord}(b_1) + \mathrm{ord}(b_2) - \mathrm{ord}(a)$$

Let $q = \max\{\mathrm{ord}(a), \mathrm{ord}(c)\}$ the change $\delta(z)$ at Z satisfies

$$\mathrm{ord}\Delta(z) \geq p - q$$