Let $\pi = \{x_{i,j}\}$, be the starting plane of indeterminates, and let $R = Z[x_{i,j}]$ be the polynomial ring with these indeterminates. For each $n \geq 1$, let $x_{i,j}(n)$ be the $n \times n$ minor with $x_{i,j}$ in the upper left corner. The $x_{i,j}(n)$ satisfy the Dodgson recursion:

For $n \geq 3$,

$$x_{i+1,j+1}(n - 2) \cdot x_{i,j}(n) = x_{i,j}(n - 1) \cdot x_{i+1,j+1}(n - 1) - x_{i,j+1}(n - 1) \cdot x_{i+1,j}(n - 1)$$

The position of $x_{i,j}(n)$ will be designated $(i, j, n)$. **Claim:** $E_{0,0}$ is in $A$

**Proof:** Let $Y = D_{0,0}D_{1,1} - D_{0,1}D_{1,0}$. Thus $E_{0,0} = \frac{Y}{C_{1,1}}$. Multiply $Y$ by $b_{1,1}$, to get

$$b_{1,1}Y = D_{1,1}(c_{0,0}C_{1,1} - c_{0,1}c_{1,0}) - b_{1,1}D_{0,1}D_{1,0}$$

$$= C_{1,1}X_1 - U$$

where $X_1 = D_{1,1}c_{0,0}$, and $U = c_{0,1}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}D_{1,0}$.

$$b_{2,1}U = b_{2,1}c_{0,1}c_{1,0}D_{1,1} + b_{2,1}b_{1,1}D_{0,1}D_{1,0}$$

$$= b_{2,1}c_{1,0}c_{1,0}D_{1,1} + b_{1,1}D_{0,1}(c_{1,0}c_{2,1} - C_{1,1}c_{2,0})$$

$$= c_{1,0}V - C_{1,1}X_2$$

where $X_2 = b_{1,1}D_{0,1}c_{2,0}$ and $V = b_{2,1}D_{1,1}c_{0,1} + b_{1,1}D_{0,1}c_{2,1}$

$$b_{2,2}V = b_{2,2}b_{2,1}D_{1,1}c_{0,1} + b_{2,2}b_{1,1}D_{0,1}c_{2,1}$$

$$= b_{2,1}c_{0,1}(C_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + b_{2,2}b_{1,1}D_{0,1}c_{2,1}$$

$$= C_{1,1}X_3 + c_{2,1}W$$

where $X_3 = b_{2,1}c_{0,1}c_{2,2}$ and $W = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}D_{0,1}$

$$b_{1,2}W = -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}b_{1,2}D_{0,1}$$

$$= -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}(c_{0,1}c_{1,2} - c_{0,2}C_{1,1}))$$

$$= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(-b_{1,2}b_{2,1} + b_{1,1}b_{2,2})$$

$$= -b_{1,1}b_{2,2}c_{0,2}C_{1,1} + c_{0,1}c_{1,2}(a_{2,2}C_{1,1})$$

$$= X_4C_{1,1}$$
where

\[ X_4 = a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2} \]
\[ = a_{2,2}c_{0,1}c_{1,2} - a_{2,2}c_{1,1}c_{0,2} - b_{1,2}b_{2,1}c_{0,2} \]
\[ = a_{2,2}b_{1,2}d_{0,1} - b_{1,2}b_{2,1}c_{0,2} \]

\[ FY = b_{1,2}b_{2,2}b_{2,1}C_{1,1}X_1 \]
\[ - b_{1,2}b_{2,2}C_{1,1}X_2 \]
\[ - b_{1,2}c_{1,0}C_{1,1}X_3 \]
\[ - c_{1,0}c_{2,1}C_{1,1}X_4 \]

\[ = C_{1,1}(b_{1,2}b_{2,2}b_{2,1}D_{1,1}c_{0,0} \]
\[ - b_{1,2}b_{2,2}b_{1,1}c_{2,0}D_{0,1} \]
\[ - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \]
\[ - c_{1,0}c_{2,1}X_4) \]

\[ = C_{1,1}(b_{1,2}b_{2,2}b_{2,1}c_{0,0}(d_{1,1}) \]
\[ - b_{1,2}b_{2,2}b_{1,1}c_{2,0}(d_{0,1}) \]
\[ - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \]

A calculation (reversing the steps above), shows that

\[ FY = C_{1,1}(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} \]
\[ - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \]
\[ = C_{1,1}x \]

where

\[ x = b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} \]
\[ + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \]
(1) If we set \( \epsilon \) to 0, then \( Y \) becomes
\[
y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0},
\]
and we have
\[
b_{1,1}b_{2,2}b_{1,2}b_{2,1}y = c_{1,1}x
\]
Each of these expressions \((b_{1,1}b_{2,2}b_{1,2}b_{2,1}, y, c_{1,1}, x)\) is a polynomial in the original ring \( Z[x_{i,j}] \), and \( c_{1,1} \) is indecomposable. Therefore
\[
\frac{x}{F} = \frac{y}{c_{1,1}} = \epsilon_{0,0}
\]

(2) The expression for \( Z \) must now be divided by \( C_{1,1} \) The first and second terms are easy:
\[
b_{1,2}b_{2,1}c_{0,0}c_{2,2}\delta = c_{0,0}c_{2,2} - \frac{\epsilon}{a_{2,2}}
\]
\[
b_{1,2}b_{2,1}c_{2,0}c_{2,0}\delta = c_{2,0}c_{0,2} - \frac{\epsilon}{a_{2,2}}
\]

(3) Let \( P = \left( c_{1,0}c_{2,1}c_{0,1}c_{1,2} - C_{1,1}c_{1,1}c_{2,0}c_{0,2} \right) a_{2,2}\delta \). Using
\[
c_{1,0}c_{2,1} = b_{2,1}D_{1,0} + c_{2,0}C_{1,1}
\]
\[
c_{0,1}c_{1,2} = b_{1,2}D_{0,1} + c_{0,2}C_{1,1},
\]
we have
\[
P = \left( (b_{2,1}D_{1,0} + c_{2,0}C_{1,1}) \cdot (b_{1,2}D_{0,1} + c_{0,2}C_{1,1}) - C_{1,1}c_{1,1}c_{2,0}c_{0,2} \right) a_{2,2}\delta
\]
\[
= b_{2,1}b_{1,2}D_{1,0}D_{0,1}a_{2,2}\delta
\]
\[
+ b_{2,1}D_{1,0}c_{0,2}C_{1,1}a_{2,2}\delta
\]
\[
+ c_{2,0}b_{1,2}D_{0,1}C_{1,1}a_{2,2}\delta
\]
\[
+ \left( c_{2,0}C_{1,1}c_{0,2}C_{1,1} - C_{1,1}c_{1,1}c_{2,0}c_{0,2} \right) a_{2,2}\delta
\]
\[
= D_{1,0}D_{0,1}\epsilon
\]
\[
+ D_{1,0}c_{0,2}C_{1,1} \frac{\epsilon}{b_{1,2}}
\]
\[
+ c_{2,0}D_{0,1}C_{1,1} \frac{\epsilon}{b_{2,1}}
\]
\[
+ C_{1,1}c_{2,0}c_{0,2} \cdot (C_{1,1} - c_{1,1}) \cdot \frac{\epsilon}{b_{1,2}b_{2,1}}
\]
(3a) After dividing by $C_{1,1}$, the first term becomes $D_{1,0}D_{0,1}\frac{\epsilon}{C_{1,1}}$, which is in $A$.

(After expanding the definitions of $D_{1,0}, D_{0,1}$, this will involve some quadratic and cubic terms in $\epsilon$.)

(3b) After dividing by $C_{1,1}$, the second term becomes $D_{1,0}c_{0,2}\frac{\epsilon}{b_{1,2}}$, which is in $A$.

(3c) After dividing by $C_{1,1}$, the third term becomes $c_{2,0}D_{0,1}\frac{\epsilon}{b_{2,1}}$, which is in $A$.

(3d) After dividing by $C_{1,1}$, and using $C_{1,1} - c_{1,1} = \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}}$, the fourth term becomes

$$c_{2,0}c_{0,2} \cdot \frac{b_{1,1}b_{2,2}\epsilon}{a_{2,2}} \cdot \frac{\epsilon}{b_{1,2}b_{2,1}}.$$  

Using the identity:

$$\frac{b_{1,1}b_{2,2}\epsilon}{b_{1,2}a_{2,2}} = c_{1,1} \frac{\epsilon}{b_{1,2}} + b_{2,1} \frac{\epsilon}{a_{2,2}}$$

this becomes

$$c_{2,0}c_{0,2} \cdot \left( c_{1,1} \frac{\epsilon}{b_{1,2}} + b_{2,1} \frac{\epsilon}{a_{2,2}} \right) \frac{\epsilon}{b_{2,1}}$$

which is in $A$. (Some terms quadratic in $\epsilon$ appear.) The net result is that $\frac{Y}{C_{1,1}}$ is in the ring $R[\frac{\epsilon}{a_{2,2}}, \frac{\epsilon}{b_{1,2}}, \frac{\epsilon}{b_{2,1}}, \frac{\epsilon}{C_{1,1}}]$.  

*****************************************************************************

To summarize and clarify the situation, this calculation shows that

$$FY = C_{1,1}x + C_{1,1}FZ_1 + FD_{1,0}D_{0,1} \cdot \epsilon,$$

where

(1) $x$ is the polynomial in $Z[x_{i,j}]$ that results if $\epsilon$ is set to 0, and $\frac{x}{F} = e_{0,0}$.

(2) $Z_1$ is demonstrably in $R[\frac{\epsilon}{a_{2,2}}, \frac{\epsilon}{b_{1,2}}, \frac{\epsilon}{b_{2,1}}]$, and

(3) $\frac{\epsilon}{C_{1,1}}$ is the fraction that is adjoined to $A$ at this stage. Therefore
\[ E_{0,0} = \frac{Y}{C_{1,1}} = e_{0,0} + Z_1 + D_{1,0}D_{0,1} \frac{\epsilon}{C_{1,1}} \text{ is in } A. \]

Start with \( y = d_{0,0}d_{1,1} - d_{0,1}d_{1,0} \). Multiply by \( b_{1,1} \), to get

\[
b_{1,1}y = d_{1,1}(c_{0,0}c_{1,1} - c_{0,1}c_{1,0}) - b_{1,1}d_{0,1}d_{1,0} = c_{1,1}x_1 - u
\]

where \( x_1 = d_{1,1}c_{0,0} \), and \( u = c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}d_{1,0} \).

\[
b_{2,1}u = b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{2,1}b_{1,1}d_{0,1}d_{1,0} = b_{2,1}c_{0,1}c_{1,0}d_{1,1} + b_{1,1}d_{0,1}(c_{1,0}c_{2,1} - c_{1,1}c_{2,0}) = c_{1,0}v + c_{1,1}x_2
\]

where \( x_2 = b_{1,1}d_{0,1}c_{2,0} \) and \( v = b_{2,1}d_{1,1}c_{0,1} + b_{1,1}d_{0,1}c_{2,1} \)

\[
b_{2,2}v = b_{2,2}b_{2,1}d_{1,1}c_{0,1} + b_{2,2}b_{1,1}d_{0,1}c_{2,1} = b_{2,1}{c}_{1,1}c_{2,2} - c_{1,2}c_{2,1}) + b_{2,2}b_{1,1}d_{0,1}c_{2,1} = c_{1,1}x_3 + c_{2,1}w
\]

where \( x_3 = b_{2,1}c_{0,1}c_{2,2} \) and \( w = -b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}d_{0,1} \)

\[
b_{1,2}w = -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}b_{1,2}d_{0,1} = -b_{1,2}b_{2,1}c_{0,1}c_{1,2} + b_{2,2}b_{1,1}(c_{0,1}c_{1,2} - c_{0,2}c_{1,1})) = -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}(-b_{2,1}b_{2,1} + b_{1,1}b_{2,2}) = -b_{1,1}b_{2,2}c_{0,2}c_{1,1} + c_{0,1}c_{1,2}a_{2,2}c_{1,1} = x_4c_{1,1}
\]

where \( x_4 = -b_{1,1}b_{2,2}c_{0,2} + c_{0,1}c_{1,2}a_{2,2} \) divisible by \( b_{1,2} \) because

\[
x_4 = a_{2,2}c_{0,1}c_{1,2} - b_{1,1}b_{2,2}c_{0,2} = a_{2,2}c_{0,1}c_{1,2} - a_{2,2}c_{1,1}c_{0,2} - b_{1,2}b_{2,1}c_{0,2} = a_{2,2}b_{1,2}d_{0,1} - b_{1,2}b_{2,1}c_{0,2}
\]
\[ F_y = b_{1,2}b_{2,2}b_{2,1}c_{1,1}x_1 - b_{1,2}b_{2,2}c_{1,1}x_2 - b_{1,2}c_{1,0}c_{1,1}x_3 - c_{1,0}c_{2,1}c_{1,1}x_4 \]
\[ = c_{1,1}(b_{1,2}b_{2,2}b_{2,1}d_{1,1}c_{0,0} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}(b_{2,1}c_{0,1}c_{2,2} - c_{1,0}c_{2,1}x_4)) \]
\[ = c_{1,1}(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2}) \]
\[ = c_{1,1}(b_{1,2}b_{2,2}b_{2,1}c_{0,0}d_{1,1} - b_{1,2}b_{2,2}b_{1,1}c_{2,0}d_{0,1} - b_{1,2}c_{1,0}b_{2,1}c_{0,1}c_{2,2} + c_{1,0}c_{2,1}b_{1,1}b_{2,2}c_{0,2} - c_{1,0}c_{2,1}c_{0,1}c_{1,2}a_{2,2} \]
The next step. Start with \( y = E_{0,0}E_{1,1} - E_{0,1}E_{1,0} \). Multiply by \( C_{1,1} \), to get

\[
C_{1,1}y = E_{1,1}(D_{0,0}D_{1,1} - D_{0,1}D_{1,0}) - C_{1,1}E_{0,1}E_{1,0}
\]

\[
= D_{1,1}x_1 - u
\]

where \( x_1 = E_{1,1}D_{0,0} \), and \( u = D_{0,1}D_{1,0}E_{1,1} + C_{1,1}E_{0,1}E_{1,0} \).

Multiply by \( c_{2,1} \), to get

\[
c_{2,1}u = c_{2,1}D_{0,1}D_{1,0}E_{1,1} + c_{2,1}C_{1,1}E_{0,1}E_{1,0}
\]

\[
= c_{2,1}D_{0,1}D_{1,0}E_{1,1} + C_{1,1}E_{0,1}(D_{1,0}D_{2,1} - D_{1,1}D_{2,0})
\]

\[
= D_{1,0}v + D_{1,1}x_2
\]

where \( x_2 = C_{1,1}E_{0,1}D_{2,0} \) and \( v = c_{2,1}E_{1,1}D_{0,1} + C_{1,1}E_{0,1}D_{2,1} \)

Multiply by \( c_{2,2} \), to get

\[
c_{2,2}v = c_{2,2}c_{2,1}E_{1,1}D_{0,1} + c_{2,2}C_{1,1}E_{0,1}D_{2,1}
\]

\[
= c_{2,1}D_{0,1}(D_{1,1}D_{2,2} - D_{1,2}D_{2,1}) + c_{2,2}C_{1,1}E_{0,1}D_{2,1}
\]

\[
= D_{1,1}x_3 + D_{2,1}w
\]

where \( x_3 = c_{2,1}D_{0,1}D_{2,2} \) and \( w = -c_{2,1}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}E_{0,1} \)

Multiply by \( c_{1,2} \), to get

\[
c_{1,2}w = -c_{1,2}c_{2,1}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}c_{1,2}E_{0,1}
\]

\[
= -c_{1,2}c_{2,1}D_{0,1}(D_{1,1}D_{2,2} - D_{1,2}D_{2,1}) + c_{2,2}C_{1,1}(D_{0,1}D_{1,2} - D_{0,2}D_{1,1})
\]

\[
= -C_{1,1}c_{2,2}D_{0,2}D_{1,1} + D_{0,1}D_{1,2}(-c_{1,2}c_{2,1} + C_{1,1}c_{2,2})
\]

\[
= -C_{1,1}c_{2,2}D_{0,2}D_{1,1} + D_{0,1}D_{1,2}b_{2,2}D_{1,1}
\]

\[
= x_4D_{1,1}
\]

where \( x_4 = -C_{1,1}c_{2,2}D_{0,2} + D_{0,1}D_{1,2}b_{2,2} \) divisible by \( c_{1,2} \) because

\[
x_4 = b_{2,2}D_{0,1}D_{1,1} - C_{1,1}c_{2,2}D_{0,2}
\]

\[
= b_{2,2}D_{0,1}D_{1,1} - b_{2,2}D_{1,1}D_{0,2} - c_{1,2}c_{2,1}D_{0,2}
\]

\[
= b_{2,2}c_{1,2}E_{0,1} - c_{1,2}c_{2,1}D_{0,2}
\]
\[ C_y = c_{1,2}c_{2,2}c_{2,1}D_{1,1}x_1 \]
\[ - c_{1,2}c_{2,2}D_{1,1}x_2 \]
\[ - c_{1,2}D_{1,0}D_{1,1}x_3 \]
\[ - D_{1,0}D_{2,1}D_{1,1}x_4 \]

\[ = D_{1,1}(c_{1,2}c_{2,2}c_{2,1}E_{1,1}D_{0,0} \]
\[ - c_{1,2}c_{2,2}C_{1,1}D_{2,0}E_{0,1} \]
\[ - c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} \]
\[ - D_{1,0}D_{2,1}x_4) \]

\[ = D_{1,1}(c_{1,2}c_{2,2}c_{2,1}D_{0,0}E_{1,1} - c_{1,2}c_{2,2}C_{1,1}D_{2,0}E_{0,1} \]
\[ - c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} + D_{1,0}D_{2,1}C_{1,1}c_{2,2}D_{0,2} \]
\[ - D_{1,0}D_{2,1}D_{0,1}D_{1,2}D_{2,2} \]

\[ = D_{1,1}(c_{1,2}c_{2,1}D_{0,0}D_{1,1}D_{2,2} - c_{1,2}c_{2,1}D_{0,0}D_{1,2}D_{2,1} \]
\[ - c_{2,2}C_{1,1}D_{2,0}D_{0,1}D_{1,2} + c_{2,2}C_{1,1}D_{2,0}D_{0,2}D_{1,1} \]
\[ - c_{1,2}D_{1,0}c_{2,1}D_{0,1}D_{1,2} + D_{1,0}D_{2,1}C_{1,1}c_{2,2}D_{0,2} - D_{1,0}D_{2,1}D_{0,1}D_{1,2}D_{2,2} \]