Inverse Problems for Electrical Networks

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Preface

This book is the result of an accumulation of work done by the authors and their students over the past twelve years. In each of the years listed, 8-10 students were brought to the University of Washington for a summer REU (Research Experience for Undergraduates) program, supported by a REU Grant from the NSF. We want to thank the NSF for its support during this period. And we want to thank the students for their enthusiasm, their dedication, and their individual contributions, without which this book would not have been possible.

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Chapter 1

Introduction

1.1 Electrical Networks

Suppose an electrical network is inside a black box as in Figure **??**. The interior of the box consists of nodes joined by conductors. The nodes are the vertices, and the conductors are the edges of a graph G.

- The *inverse problem* is to find the conductance of each edge in G from measurements of voltages and currents at the boundary nodes.

The *forward problem* assumes that the graph G and the conductance $\gamma(pq)$ of each edge pq in G are known. If a voltage is imposed at the boundary nodes, there is a function u defined throughout the network which agrees with f at the boundary nodes, and which satisfies Kirchhoff's Law at each interior node.

KIRCHHOFF'S LAW: At each interior node p, the sum of the currents from p to its neighboring nodes is 0.

This function u is called the *potential* due to f. The resulting current at the boundary nodes is called the *network response*. The linear map $\Lambda = \Lambda_\gamma$ which takes the boundary voltage f to the boundary current ϕ is called the *response map*. Λ is sometimes called the voltage-to-current map because it gives the current (i.e., the response) to any voltage imposed at the boundary nodes. The response map will be known when the potential is found for each boundary function f, and the resulting boundary current ϕ is calculated. If the standard basis is used to represent the boundary function f and the

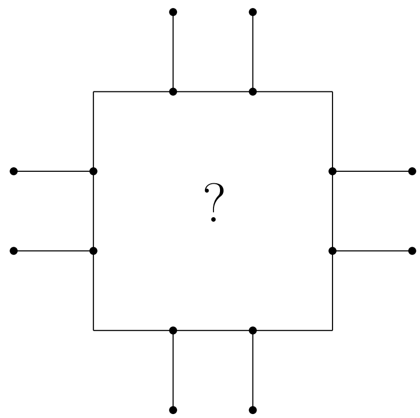


Figure 1-1: Black Box

boundary current ϕ, the response map is represented by an n by n matrix also denoted Λ, called the *response matrix*.

On the other hand, if the response map Λ_γ is given, but the conductivity function γ is unknown, the inverse problem is to use Λ_γ to calculate the conductance of each edge in G. If the graph G is unknown, then that too must be deduced from the response matrix Λ_γ.

The inverse problem as articulated (for a continuous conducting medium) by Calderon in [**?**], can be broken into four questions.

(Q1) Uniqueness: Is the map $\gamma \to \Lambda_\gamma$ one-to-one?

(Q2) Characterization: Which linear maps Λ are response maps?

(Q3) Algorithm: Is there a procedure for calculating γ from the response map Λ_γ?

(Q4) Continuity: If γ is near μ, does it follow that Λ_γ is near Λ_μ?

To these we add a fifth question.

(Q5) Can the graph G be deduced from the response matrix?

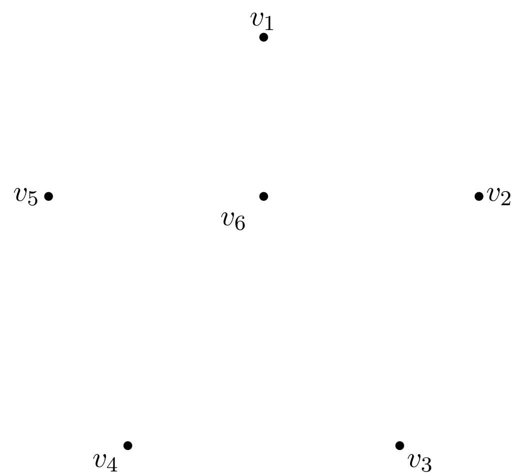


Figure 1-2: Circular planar graph G

Example 1.1 Suppose given a resistor network with five boundary nodes, one interior node and seven edges as in Figure **??**. Measurements of voltages and currents are made at the boundary nodes v_1, v_2, v_3, v_4, v_5, located on the dashed circle (not part of the network). The response matrix Λ is:

$$\Lambda = \begin{bmatrix} 2.4 & -1.2 & -0.8 & 0 & -0.4 \\ -1.2 & 7.1 & -5.6 & 0 & -0.3 \\ -0.8 & -5.6 & 7.6 & -1.0 & -0.2 \\ 0 & 0 & -1.0 & 4.0 & -3.0 \\ -0.4 & -0.3 & -0.2 & -3.0 & 3.9 \end{bmatrix} \tag{1.1}$$

The inverse problem is to calculate the conductances of each of the seven edges in G from Λ.

Returning to the general situation, the solution to the forward problem reveals some facts about the response matrix. For any resistor network $\Gamma = (G, \gamma)$ with n boundary nodes, the response matrix Λ is an n by n matrix which has the following three properties.

(1) Λ is symmetric: $\Lambda_{i,j} = \Lambda_{j,i}$

(2) The sum of the entries in each row is 0.

(3) For $i \neq j$, $\Lambda_{i,j} \leq 0$

If the graph is allowed to be arbitrary, there is an easy answer to the inverse problem. *Every* n by n matrix Λ which satisfies (1), (2) and (3) is the response matrix for a suitable conductivity on a subgraph of the complete graph with n nodes $v_1, v_2, \ldots, v_n$, as follows. For each pair (i, j) with $\Lambda_{i,j} \neq 0$, place an edge joining v_i to v_j and assign the conductance of this edge to be $\gamma(v_i v_j) = -\Lambda_{i,j}$. The response matrix for this network will be $\Lambda_\gamma = \Lambda$. The inverse problem for resistor networks becomes interesting only if a restriction is placed on the type of graph allowed in the interior of the box. A *circular planar resistor network* consists of a graph, embedded in the disc in the plane, with the boundary nodes on the boundary circle, and with a conductance assigned to each of the edges.

- This text is concerned with circular planar resistor networks.

The surprising outcome is that, for circular planar resistor networks, there is a positive answer to the five questions **(Q1) - (Q5)**. The answers involve three main techniques, which turn out to be closely related.

(I) Schur complements.

(II) Harmonic continuation.

(III) Medial graphs.

The first use of Schur complements is to obtain the response matrix from the Kirchoff matrix. More subtle is the use of Schur complements to obtain formulas (**??**) and (**??**) for calculating conductances of boundary edges and boundary spikes from the response matrix Λ. These same formulas can also be arrived at by a process called harmonic continuation. The medial graphs in Chapters **??** and **??** give even more insight into the same formulas. These three techniques are discussed briefly in the remainder of this chapter, and will be dealt with in much greater detail in the succeeding chapters.

Some important concepts concerning circular planar graphs are *path*, *connection*, *critical*, and *well-connected*. A *path* $p \leftrightarrow q$ between two boundary nodes p and q is a sequence of nodes in the interior of G whose edges join p to q. For example in the graph of Figure **??**, the path $v_1 v_6 v_2$ joins v_1 to v_2. There is no path joining v_1 to v_4 through G. If $P = (p_1, \ldots, p_k)$, and $Q = (q_1, \ldots, q_k)$ are sequences of boundary points, a k-*connection* through G, denoted $P \leftrightarrow Q$, is a set of paths $\{p_i \leftrightarrow q_i\}$ which are vertex disjoint.

In Figure **??**, there is a 2-connection through G from (v_1, v_5) to (v_2, v_4), but there is no 2-connection from (v_1, v_5) to (v_2, v_3). The set of connections (through a graph G) between pairs of sequences of boundary nodes which are in disjoint arcs of the circle, will be denoted $\pi(G)$. A graph is called *well-connected* if it has all possible connections between pairs of sequences of boundary nodes which are in disjoint arcs of the boundary circle. A graph is called *critical* if removing any edge breaks a connection. Roughly speaking, this means that every edge is essential. Every circular planar resistor network is electrically equivalent to one whose underlying graph is critical. These concepts will be examined more fully in Chapters **??** and **??**.

The matrix construction called *Schur complement* is essential for almost all the later algebraic development. If $\Gamma = (G, \gamma)$ is a resistor network, the *Kirchhoff matrix* $K = K_\Gamma$ gives the response currents $\phi = Ku$ at all nodes (interior and boundary) to a voltage defined at all the nodes of the network. In Chapter 3, the response matrix of a network is shown to be obtained by taking the Schur complement in K of a certain submatrix.

Example 1.2 Let $\Gamma = (G, \gamma)$ be the resistor network where G is the graph of Figure **??**, and the conductances of the edges are: $\gamma(v_1v_6) = 4$, $\gamma(v_2v_6) = 3$, $\gamma(v_2v_3) = 5$, $\gamma(v_3v_6) = 2$, $\gamma(v_3v_4) = 1$, $\gamma(v_4v_5) = 3$, $\gamma(v_5v_6) = 1$. The Kirchhoff matrix is

$$K = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & -4 \\ 0 & 8 & -5 & 0 & 0 & -3 \\ 0 & -5 & 8 & -1 & 0 & -2 \\ 0 & 0 & -1 & 4 & -3 & 0 \\ 0 & 0 & 0 & -3 & 4 & -1 \\ -4 & -3 & -2 & 0 & -1 & 10 \end{bmatrix}$$

In this case, the response matrix Λ_γ is the Schur complement in K of the lower right hand corner entry, which is the number 10. Λ_γ is the 5 by 5 matrix in the upper left corner obtained by row-reduction using this entry. The result is the matrix Λ which was given in equation **??**.

NOTATION: The entry at the (i, j) position of a matrix A will sometimes be referred to as $A(i; j)$, instead of the usual $A_{i,j}$. This is convenient when i and j themselves are subscripted. This notation also extends to a convenient notation for submatrices. Thus $A_\gamma(1,5;2,4)$ means the 2 by 2 submatrix of A with entries from rows 1 and 5, and columns 2 and 4. (See Chapter **??**.)

Many of the properties of the response matrix follow from its expression as a Schur complement. In particular, Theorem **??** shows that connections through the network can be determined from its response matrix. For example the statement that $\Lambda_\gamma(1;2) \neq 0$ implies that there is a path from v_1 to v_2 through G. The statement that $\Lambda_\gamma(1;4) = 0$ implies that there is no path from v_1 to v_4 through G. The statement that $\det \Lambda_\gamma(1,5;2,4) \neq 0$ implies there is a 2-connection through G from (v_1, v_5) to (v_2, v_4). The statement $\det \Lambda_\gamma(1,5;2,3) = 0$ implies there is no 2-connection through G from (v_1, v_5) to (v_2, v_3). The relation between connections through the graph and subdeterminants of the response matrix is important both theoretically, and for the numerical recovery of conductors. It is one of the cornerstones of the theory presented in this text.

There are two formulas for computing the values of conductors at the boundary of the network directly from the response matrix Λ_γ. Formula **??** gives the conductance of a boundary edge. Formula **??** gives the conductance of a boundary spike. These formulas are used in the proof that Λ_γ uniquely determines γ.

It is important to be able to construct harmonic functions (that is, potentials) on a resistor network, given various type of boundary data. This construction is made by a process called *harmonic continuation*, described in Chapter **??**. Harmonic continuation is used to show the existence of several types of harmonic functions which are the basis for the recovery algorithm of Chapter **??**. Using these functions, there is a direct way to calculate the conductors in a rectangular network, and there is a direct way to calculate the conductors in the well-connected circularly symmetric graphs G_n introduced in Chapter **??**.

Harmonic continuation is also used to show the existence of functions needed to characterize the set of response matrices for circular planar graphs. For each integer n, there is a well-connected critical graph with n boundary nodes, unique to within $Y - \triangle$ equivalence. One such graph is G_n described in Chapter **??**, and another is the graph H_n, which is $Y - \triangle$ equivalent to G_n, described in Chapter **??**. For either of these graphs, the set of response matrices is a certain set of matrices $L(n)$, which is simply described in terms of signs of subdeterminants of the response matrices. The first of the characterization theorems, Theorem **??**, shows that the set of response matrices for G_n (or equivalently, H_n) is $L(n)$.

There are three ways to adjoin an edge to a graph:

(1) Adjoin a boundary edge. An edge is added between two adjacent boundary nodes.

(2) Adjoin a boundary pendant. A boundary node is added together with an edge joining that node to an old boundary node. This increases the number of boundary nodes by one.

(3) Adjoin a boundary spike. A new boundary node is added together with an edge joining that node to an old boundary node. The old boundary is then declared to be an interior node. This does not change the number of boundary nodes.

If $\Gamma = (G, \gamma)$ is a resistor network, the effect on the response matrix of adjoining a boundary edge conductor or a boundary spike conductor is also described in Chapter **??**. By considering the reverse operations, the effect on the response matrix of removing a conductor from the network is also described. This leads to Theorem **??** and its corollaries which show that the values of the conductors in any well-connected critical graph can be recovered from the response matrix. This requires a result from Chapter **??** showing that there is always at least one boundary edge or boundary spike. The boundary edge formula (Equation **??**) or the boundary spike formula (Equation **??**) is then used to calculate the conductance. Each time an edge is removed a new graph is formed, which is critical, and its response matrix is calculated. In this way, if $\Gamma = (G, \gamma)$ is a resistor network whose underlying graph is critical, the conductances of all the edges in G can be calculated. At the conclusion of Chapter **??**, questions **(Q1)**, **(Q3)**, and **(Q4)** have been answered affirmatively for all circular planar graphs, as well as question **(Q2)** for well-connected graphs.

The answer to question **(Q2)** for arbitrary circular planar graphs requires some elementary but non-standard matrix algebra, which is presented in Section **??**. In Section **??** the set of response matrices (for arbitrary circular planar graphs) is proven to be a certain set of matrices Ω_n which is defined in terms of signs of subdeterminants. The only use made of Section **??** is to prove Theorem **??**, where it is shown that every matrix in Ω_n is the response matrix for a conductivity function on a suitable critical but not necessarily well-connected graph. This set Ω_n is of course closely related to the set $L(n)$ defined in Chapter **??**.

The answer to **(Q5)** makes use of the medial graph associated to a circular planar graph. The relation between circular planar graphs and their medial graphs is brought out in Chapter **??**. The set of $Y - \triangle$ equivalence classes of critical circular planar graphs is shown to be in 1-1 correspondence with (the equivalence) classes of medial graphs. Furthermore, (the equivalence class of) each medial graph is determined by the set S of pairs of endpoints of the medial lines on the boundary circle.

For each integer $n \geq 3$, let

- $\mathcal{G}_n$ be the set of $Y - \triangle$ equivalence classes of critical circular planar graphs with n boundary nodes.

- $\mathcal{M}_n$ be the set of equivalence classes of medial graphs arising from $\mathcal{G}_n$.

- $\mathcal{S}_n$ be the set of $\{S\}$ where each $S = \{x_i, y_i\}$, is the set of endpoints of the geodesics arising from a graph M in $\mathcal{M}_n$.

- $\diamond_n$ be the set of $\{\pi\}$, where each π is the set of connections for a graph G in $\mathcal{G}_n$.

The reason medial graphs are useful in studying circular planar graphs is that there are natural 1-1 correspondences between these four sets.

$$\mathcal{G}_n \approx \mathcal{M}_n \approx \mathcal{S}_n \approx \diamond_n$$

Chapter **??** goes into detail on the relations between a graph G, its medial graph $\mathcal{M}$, the set S of pairs of numbers which define the endpoints of the geodesics, and the set of connections $\pi(G)$ through G. The connections $\pi(G)$ are obtained from subdeterminants of the response matrices. Theorem **??** which summarizes this material, is paraphrased as follows.

(1) If a matrix A is given which is in the algebraically defined set Ω_n, Proposition **??** shows how to construct a medial graph, and from it a circular planar graph G.

(2) The formulas of Chapter **??** can be used to calculate conductances on G so that the response matrix is A.

Thus all five of the questions **(Q1) - (Q5)** for inverse problems for circular planar resistor networks have been answered.

Layered networks are discussed in Chapter **??**. These are circularly symmetric graphs, where the conductivity is constant on the layers. We present a theorem (from David Ingerman's thesis, [**?**]) which characterizes the response matrices for such networks. It is believed, but not yet established, that the recovery of conductances is much better for these graphs than for arbitrary circular planar resistor networks.

1.2 Other Topics

There are several topics related to inverse problems for electrical networks, which are not covered here. Among these are the following.

1. Duality. To each circular planar graph G there is a circular planar graph $G^{\perp}$ called the dual graph. This is mentioned briefly in conjunction with the 2-coloring of the regions in the disc defined by medial graph $\mathcal{M}$. If the black regions are the nodes of G, the white regions are the nodes of $G^{\perp}$. To each circular planar resistor network $\Gamma = (G, \gamma)$ there is a dual network $\Gamma^{\perp} = (G^{\perp}, \gamma^{\perp})$. The conductance of each edge in $G^{\perp}$ is the reciprocal of the conductance of the edge in G which it crosses. This is gone into more fully in [**?**], where the relation between the "voltage-to-current" map and the "current-to-voltage" map is also discussed.

2. Markov Chains. A resistor network Γ gives rise to a reversible Markov chain, as described in [**?**]. It is possible that the techniques in this text can be used to handle inverse problems for reversible Markov chains, especially ones for which the graph is circular planar. As far as we know at the present time, this is unexplored.

3. Continuous Media. The inverse problem for resistor networks is analogous to the inverse problem for a continuous conducting medium. Suppose R is a compact region, with boundary, in Euclidean space, with a conductivity function γ which is positive on R. If a boundary voltage f is given, the solution to the Dirichlet problem is a potential u defined throughout R, which agrees with f on the boundary of R, and which satisfies the conductivity equation, inside R. That is,

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \text{ inside R} \\ u &= f \text{ on the boundary of } R \end{aligned}$$

The boundary current due to the potential u is the function $\phi = \gamma \dfrac{\partial u}{\partial n}$ where n is the unit inward normal to the boundary of R. In the continuous case, the response map is sometimes called the Dirichlet-to-Neumann map because the function f is Dirichlet boundary data, and the function ϕ is Neumann boundary data for the conductivity equation. A circular planar resistor network is a discretization of a continuous conducting medium occupying a bounded region R in the plane, and Kirchhoff's Law is a discretization of the conductivity equation. A great deal is known about the uniqueness **(Q1)** and the continuity question **(Q4)**. Research is currently focused on the reconstruction algorithm **(Q3)**. The characterization problem for continuous media is very much open. Some of the properties characterizing response matrices for resistor networks carry over to the continuous case, but a complete characterization has not yet been attained. An up-to-date source for inverse problems is [**?**].

4. The Inverse Problem is Ill-posed. It should be pointed out that even though the answer to **(Q4)** is positive in that γ depends continuously on Λ_γ (as a rational function in the entries of Λ_γ), the calculations of the conductances is extremely sensitive to small errors in Λ. Thus, if the response matrix Λ_γ is known only approximately, the recovery of the conductances might be very poor, even meaningless. That is, the calculation might give negative values for the conductances. Roughly speaking, one digit of accuracy is lost for each additional layer in the network. In this sense, the general inverse problem for circular planar resistor networks is ill-posed.

Chapter 2

Circular Planar Graphs

2.1 Connections

A *graph with boundary* is a triple $G = (V, V_B, E)$, where V is the set of nodes and E is the set of edges for a finite graph, and V_B is a nonempty subset of V called the set of boundary nodes. The set $I = V - V_B$ is called the set of *interior nodes*. G is allowed to have multiple edges (that is, more than one edge joining two nodes) or loops (a loop is an edge joining a node to itself) or pendants (a pendant is an edge with one endpoint of valence one. Examples are given in Figures **??**, **??** and **??**.

A *circular planar graph* is a graph G with boundary which is embedded in a disc D in the plane so that the boundary nodes lie on the circle C which bounds D, and the rest of G is in the interior of D. The boundary nodes are labeled $v_1, \ldots, v_n$ in clockwise order around C. Figure **??** shows a circular planar graph with 6 boundary nodes, 2 interior nodes (8 nodes altogether), and 9 edges. The dashed circle indicates the boundary circle.

If p and q are distinct boundary nodes, a *path* β *from* p to q *through* G, consists of a sequence of edges: $e_0 = pr_1, \;\; e_1 = r_1r_2, \;\; \ldots, e_{h-1} = r_{h-1}r_h, \;\; e_h = r_hq$ such that $r_1, r_2, \ldots, r_h$ are *distinct interior* nodes of G. An edge pq between two distinct boundary nodes p and q is allowed as a path from p to q through G. If each of the edges is uniquely specified by its endpoints, or if it only matters which interior vertices are along the path β, the path may be written as:

$$\beta = pr_1r_2 \cdots r_hq$$

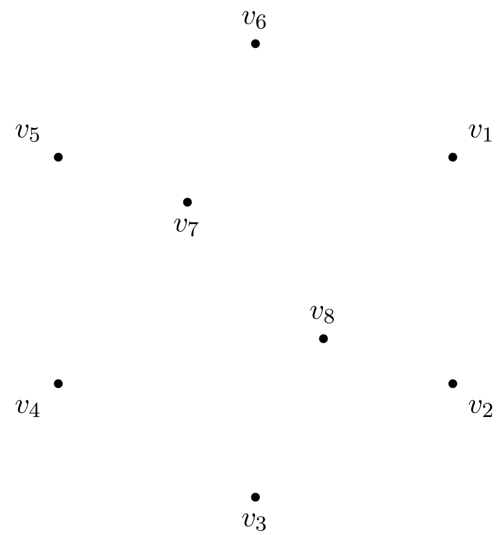


Figure 2-1: Circular planar graph G

The existence of a path from p to q through G is sometimes indicated simply by $p \leftrightarrow q$.

Example 2.1 In the graph G of Figure **??**, starting from v_1, there are the following paths originating from v_1:

$$\begin{aligned}
v_1 \leftrightarrow v_2 : \beta_1 &= v_1 v_7 v_8 v_2 \\
v_1 \leftrightarrow v_3 : \beta_2 &= v_1 v_7 v_8 v_3 \\
v_1 \leftrightarrow v_5 : \beta_3 &= v_1 v_7 v_5 \\
v_1 \leftrightarrow v_6 : \beta_4 &= v_1 v_6
\end{aligned}$$

but there is no path from v_1 to v_4 through G. The existence of paths is reflexive; that is, a path from p to q implies that there is a path from q to p, namely the same vertices taken in the opposite order. But paths are not transitive, as the above example shows: there is a path from v_1 to v_3 and also a path from v_3 to v_4, but there is no path from v_1 to v_4.

Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are two sequences of boundary nodes. P and Q are said to be *connected through* G if there is a permutation τ of the indices $1, \ldots, k$, and k disjoint paths $\alpha_1, \ldots, \alpha_k$ in G, such that for each i, the path α_i starts at p_i, ends at $q_{\tau(i)}$, and passes through no other boundary nodes. To say that the paths $\alpha_1, \ldots, \alpha_k$ are disjoint means that if $i \neq j$, then α_i and α_j have no vertex in common. The set $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ is called a *k-connection* from P to Q. The existence of a connection is denoted $P \leftrightarrow Q$, similar to the notation $p \leftrightarrow q$ for the existence of a path between two boundary vertices p and q. A path which joins one boundary node to another boundary node is a 1-connection.

A sequence $r_1, r_2, \ldots, r_m$ of distinct points on the boundary circle C is said to be in *circular order* around C if $\widehat{r_1 r_m}$ is an arc of C, the points $r_2, \ldots, r_{m-1}$ are in the arc $\widehat{r_1 r_m}$ and

$$r_1 < r_2 < \ldots < r_m$$

in the linear order induced by the angles of the arc, measured clockwise from r_1. A pair of sequences of boundary nodes $(P; Q) = (p_1, \ldots, p_k;\ q_1, \ldots, q_k)$ such that the sequence $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order is called a *circular pair*. If $(P; Q)$ is a circular pair, the vertices in Q are in the reverse order of those in P on the boundary circle C. In this case, any connection $P \leftrightarrow Q$ must connect p_i to q_i for $i = 1, \ldots, k$.

- If G is a circular planar graph, $\pi(G)$ will denote the set of connections $P \leftrightarrow Q$ through G where the $(P; Q)$ are a circular pairs.

Remark 2.1 Suppose G is a circular planar graph, and P and Q are two sequence that are in disjoint arcs of the circle. Any connection $P \leftrightarrow Q$ can be considered to be a connection $P' \leftrightarrow Q'$ of a circular pair $(P'; Q')$ where P' and Q' are suitable permutations of P and Q. Because of this, $\pi(G)$ is defined to be the set of circular pairs $(P; Q)$ that are connected through G.

Example 2.2 In the graph G of Figure **??**, let $P = (v_1, v_2)$ and $Q = (v_5, v_3)$ and $R = (v_6, v_1)$. Then

(i) $(P; Q)$ is a circular pair that is 2-connected. The paths are:

$$\begin{aligned} v_1 \leftrightarrow v_5 &\ :\ v_1 v_7 v_5 \\ v_2 \leftrightarrow v_3 &\ :\ v_2 v_8 v_3 \end{aligned}$$

(a) Y (b) $\triangle$

Figure 2-2: Y and $\triangle$

(ii) $(R;Q)$ is a circular pair that is not 2-connected. There is a 1-connection from v_6 to v_5; there is a 1-connection from v_1 to v_3. The paths are:

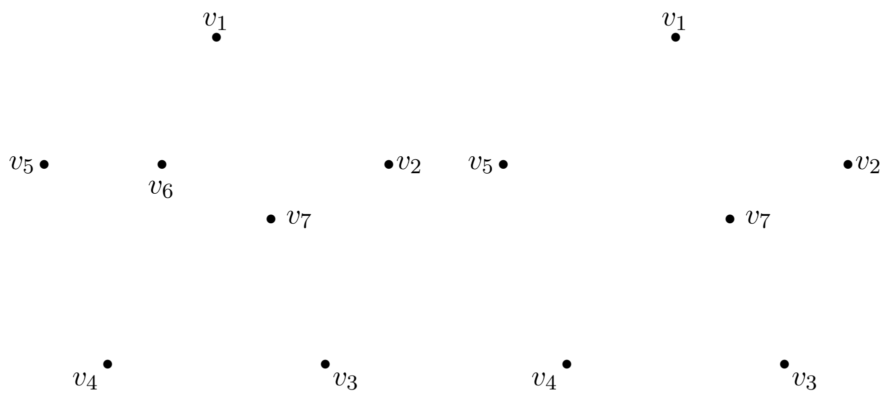
$$\begin{aligned} v_6 \leftrightarrow v_5 &\quad : \quad v_6v_7v_5 \\ v_1 \leftrightarrow v_3 &\quad : \quad v_1v_7v_8v_3 \end{aligned}$$

but there is no 2-connection from $R = (v_6, v_1)$ to $R = (v_5, v_3)$, because the the paths $v_6 \leftrightarrow v_5$ and $v_1 \leftrightarrow v_3$ are not disjoint (each path uses v_7).

(iii) There is a 3-connection $(v_1, v_2, v_3) \leftrightarrow (v_6, v_5, v_4)$. but there is no 3-connection from (v_2, v_3, v_4) to (v_1, v_6, v_5) nor from (v_3, v_4, v_5) to (v_6, v_1, v_2).

2.2 $Y - \triangle$ Transformations

Suppose a circular planar graph G contains a configuration such as that of Figure **??**a. Within the region indicated by the dotted circle, there are four vertices, p, q, r and w, and three edges pw, qw and rw; the vertex w is not a boundary node of G and there are no other edges incident to w. There may be any number of edges incident to p, q or r. Such a configuration is called a Y in the graph G. A $Y - \triangle$ *transformation* in G consists in eliminating the vertex w, deleting the edges pw, qw, rw, and inserting edges pq, qr and rp to form the $\triangle$ as in Figure **??**b. No other vertices or edges are affected.



(a) Graph G with Y at v_6 (b) Graph G' with $\triangle v_1v_7v_5$

Figure 2-3: $Y-\triangle$ transformation

Replacing the configuration of Figure **??**b with that of Figure **??**a reverses the process, and is called a $\triangle - Y$ transformation

If G and G' are two circular planar graphs and there is a sequence of graphs:

$$G = G_0, G_1, \dots G_k = G'$$

where each G_{i+1} is obtained from G_i by either a $Y-\triangle$ or a $\triangle - Y$ transformation, G and G' are said to be $Y-\triangle$ *equivalent*. The graph G' in Figure **??**b is $Y-\triangle$ equivalent to the graph G of Figure **??**a: the Y with vertex at v_6 in G has been replaced by a $\triangle$ with sides v_1v_7, v_7v_5 and v_5v_1 in the graph G'.

Refer back to Figure **??**. Suppose a graph G is transformed into a graph G' by a $Y-\triangle$ transformation, where the Y of Figure **??**a in G is replaced by the triangle of Figure **??**b in G'. Suppose α and β are disjoint paths in G where α passes through p and β passes through edges rw and wq. There are corresponding paths α' and β' in G', where $\alpha' = \alpha$ and β' is the same as β except that the two edges rw and wq are replaced by the single edge rq. That is, if

$$\begin{aligned} \alpha &= a_1 \dots p \dots a_2 \\ \beta &= b_1 \dots rwq \dots b_2 \end{aligned}$$

then

$$\begin{aligned}\alpha' &= a_1 \dots p \dots a_2 \\ \beta' &= b_1 \dots rq \dots b_2\end{aligned}$$

It follows that if α is any connection $P \leftrightarrow Q$ through G, then there is a corresponding connection $P \leftrightarrow Q$ through G'. These observations are summarized in the following important theorem.

Theorem 2.1 *Suppose G and G' are two circular planar graphs which are $Y-\Delta$ equivalent. Then the set of connections in G is in $1-1$ correspondence with the set of connections in G'.*

Example 2.3 In Figure **??**a, let $P = (v_1, v_2)$ and $Q = (v_5, v_3)$. There is a 2-connection (α_1, α_2) from P to Q through G, where

$$\begin{aligned}v_1 \leftrightarrow v_5 : \alpha_1 = v_1v_6v_5 \\ v_2 \leftrightarrow v_3\alpha_2 = v_2v_7v_3\end{aligned}$$

There is still a 2-connection from P to Q through G' of Figure **??**, where

$$\begin{aligned}v_1 \leftrightarrow v_5 : \alpha_1' = v_1v_5 \\ v_2 \leftrightarrow v_3\alpha_2' = v_2v_7v_3\end{aligned}$$

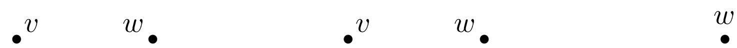
2.3 Edge Removal

There are two ways to *remove* an edge from a graph:

(1) By *deleting* an edge such as vw in G as in Figure **??**a. Deleting edge vw replaces this configuration with that of Figure **??**b. An edge joining two boundary nodes of G may be deleted.

(2) By *contracting* an edge to one of its endpoints. The edge vw in Figure **??**a has been contracted to a single node w in Figure **??**c. An edge joining two boundary nodes is not allowed to be contracted to a single node.

Suppose G is a circular planar graph and removal of an edge e, either by deletion or contraction, results in a graph G'. Let $P = (p_1, \dots, p_k)$ and



(a) edge vw (b) vw deleted (c) vw contracted

Figure 2-4:

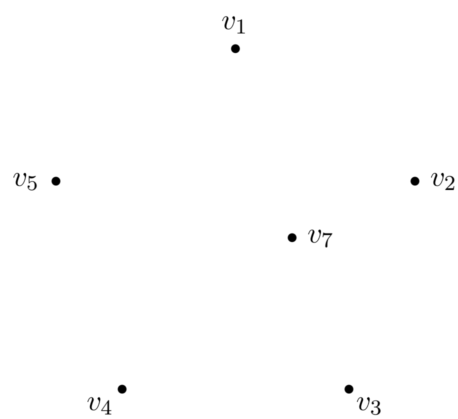


Figure 2-5: Edge removed

$Q = (q_1, \ldots, q_k)$ be two sequences of boundary nodes of G, which form a circular pair. We say that removing edge e *breaks the connection* from P to Q if there is a k-connection from P to Q through G, but there is not a k-connection from P to Q through G'. A graph G is called *critical* if the removal of any edge breaks some connection through G. Contracting v_6v_7 to a single node v_7 in the graph G of Figure **??**a results in the graph of Figure **??**. Notice also that deleting the edge v_1v_5 of Figure **??**b results in the same graph in Figure **??**. In either case, the 2-connection from $P = (v_1, v_2)$ to $Q = (v_5, v_3)$ has been broken.

Lemma 2.2 *Suppose G and G' are two circular planar graphs which are $Y - \Delta$ equivalent. Then G is critical if and only if G' is critical.*

Proof: Suppose G is transformed into G' where the Y of Figure **??**a is replaced by the triangle of Figure **??**b. Assume that G is not critical. There are three cases to consider.

(1) Suppose e is an edge in G which is not pw, qw, or rw and e can be removed without breaking a connection through G. Removal of the same edge in G' breaks no connection through G'.

(2) Suppose deletion of pw breaks no connection through G. Deletion of pr breaks no connection through G'.

(3) Suppose contraction of pw breaks no connection through G. Deletion of rq breaks no connection through G'.

Assume that G' is not critical. Again there are three cases.

(4) Suppose e is an edge in G' which is not pq, qr, or rp and e can be removed without breaking a connection through G'. Removal of the same edge in G breaks no connection through G.

(5) Suppose deletion of rq breaks no connection through G'. Contraction of pw breaks no connection through G.

(6) Suppose contraction of rq breaks no connection through G'. Contraction of rw breaks no connection through G.

The easiest way to see if a circular planar graph G is critical is to see if the medial graph $\mathcal{M}$ is lens-free, and use Proposition **??**. The graph G in Figure **??** is critical; by Lemma **??**, any graph $Y - \triangle$ to G is also critical.

2.4 Trivial Modifications

There are four simplifications of a graph that do not change any connections, and may be considered *trivial modifications*. These modifications are indicated in Figures **??**, **??**, **??** and **??**.

(1) Suppose G has a pair of edges $e = uv$ and $f = vw$ in series, as in Figure **??**a. Assume that v is an interior node of G, and that there are no other edges incident to v. A trivial modification of G consists in replacing the configuration of Figure **??**a with that of Figure **??**b.

(2) Suppose G has a pair of edges $e = uv$ and $f = uv$ in parallel, as in Figure **??**a. The configuration of Figure **??**a may be replaced by that of Figure **??**b.

(a) Edges in Series (b) Single edge

Figure 2-6: Series edges replaced by single edge

(a) Parallel (b) Single edge

Figure 2-7: Parallel edges replaced by single edge

(a) (b)

Figure 2-8: Pendant and Pendant removed

(a) Loop (b) Removed

Figure 2-9: Loop and Loop removed

(3) Suppose G has an interior pendant as in Figure **??**e. That is, v is an interior node of G, and there are no other edges incident to v. The configuration of Figure **??**a may be replaced by that of Figure **??**b.

(4) Suppose G has an interior loop $e = vv$, that is e is an edge for which the initial and final vertices are the same, as in Figure **??**a. The configuration of Figure **??**a may be replaced by that of Figure **??**b.

Each of the trivial modifications (1) to (4) does not break (or add) any connection through G. Therefore a graph that contains any of these configurations cannot be critical. In a later chapter, we will show that a sequence of trivial modifications and $Y - \triangle$ transformations, will transform G into a critical graph G' which has the same connections as G. In fact, G can be transformed into a critical graph G' by a sequence of trivial modifications only; the $Y - \triangle$ transformations are not needed. Anticipating the definitions of Chapter **??**, if $\Gamma = (G, \gamma)$ is an electrical network, then the conductors in G' can be chosen so that Γ and Γ' are electrically equivalent. This means that the response matrix for Γ is the same as the response matrix for Γ'.

2.5 Well-connected Graphs

A circular planar graph G is called *well-connected* if for every circular pair $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ of sequences of boundary nodes, there is a k-connection from P to Q. For each integer $n \geq 3$, we will describe a specific graph G_n with n boundary nodes, which is both well-connected and critical. In Chapter **??**, all the well-connected critical graphs with n boundary nodes, for a fixed value of n, will be shown to be $Y - \triangle$ equivalent.

For each integer $n \geq 3$, the nodes of the graph G_n are the points of intersection of n radial lines and some circles centered at the origin. The n rays ρ_0, ρ_1, $\ldots$, ρ_{n-1} originate from the origin, and are at angles θ_0, θ_1, $\ldots$, θ_{n-1} measured clockwise from the first ray ρ_0, where $0 = \theta_0 < \theta_1 < \ldots < \theta_{n-1} < 2\pi$. The circles have radii r_i with $0 < r_1 < \ldots < r_i < \ldots$. For convenience, (i, j) will represent the point which is the intersection of the circle of radius r_i with ray ρ_j. All the points $(0, j)$ are identified to the single point $(0, 0)$. If it occurs, the point $(i, j + n)$ is to be identified with the point (i, j); in particular (i, n) is the same point as $(i, 0)$.

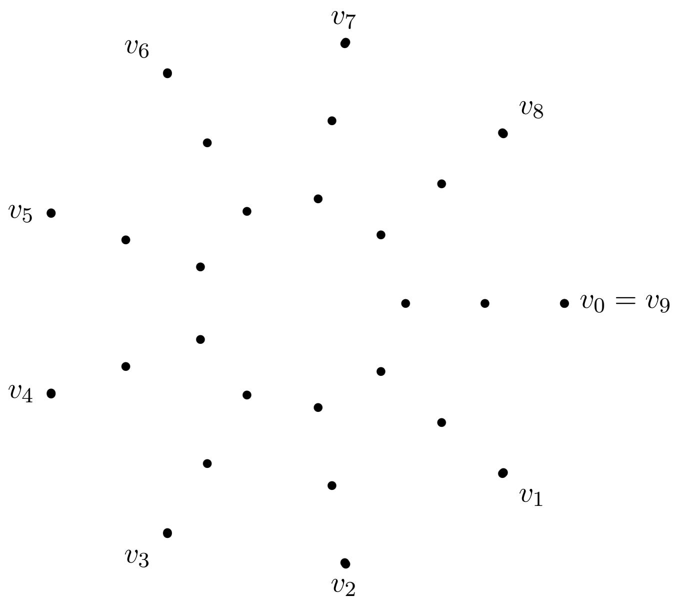


Figure 2-10: Graph G_9

Because the indexing is different for each of the four cases of n mod 4, the well-connected critical graphs G_n require four separate descriptions depending on n mod 4.

(1) Let $n = 4m + 1$. The boundary circle is the circle of radius $m + 1$, centered at $(0, 0)$. The nodes of G_{4m+1} are the points (i, j) for integers i and j with $0 < i \le m + 1$ and $1 \le j \le 4m + 1$. The radial edges are the radial line segments joining (i, j) to $(i + 1, j)$ for each $0 < i \le m$ and each $1 \le j \le 4m + 1$. The circular edges are the circular arcs joining (i, j) to $(i, j + 1)$ for each $1 \le i \le m$ and each $1 \le j \le 4m + 1$. The graph G_{4m+1} has $2m(4m + 1)$ edges and $(m - 1)(4m + 1)$ nodes. The boundary nodes of G_{4m+1} are the points $v_j = (m + 1, j)$, for $j = 1, \ldots, 4m + 1$, with the convention that $v_0 = v_{4m+1}$. The graph G_9 is shown in Figure **??**.

(2) Let $n = 4m+2$. In this case the boundary "circle" is only a topological circle (i.e., the homeomorph of a circle) to be described later. The nodes of G_{4m+2} are the points (h, j) for integers h and j with $0 \le h \le m$ and

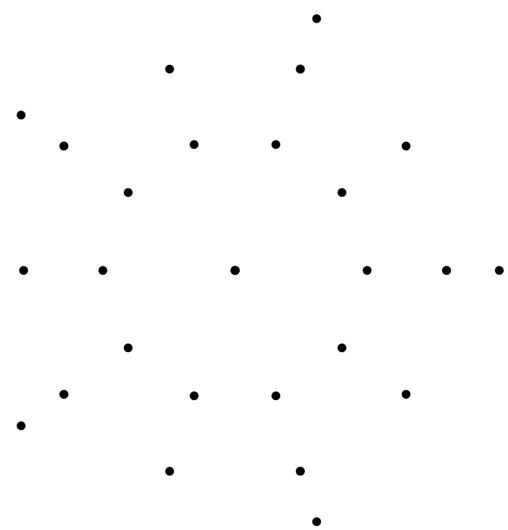


Figure 2-11: Graph G_{10}

$0 \le j \le 4m+2$. In addition, there are nodes $(m+1, j)$ for *even* values of j with $1 \le j \le 4m+2$. The radial edges are the radial line segments joining (h, j) to $(h+1, j)$ for each $0 \le h \le m-1$ and each $1 \le j \le 4m+2$, and also the radial line segments joining (m, j) to $(m+1, j)$ for each *even* value of j with $1 \le j \le 4m+2$. The circular edges are the circular arcs joining (h, j) to $(h, j+1)$ for each $1 \le h \le m$ and each value of $1 \le j \le 4m+2$. In this case, the boundary nodes are the points $(m+1, j)$ for even values of j, and points (m, j) for odd values of j. There are $(2m+1)(4m+1)$ edges in G_{4m+2} and there are $(2m+1)^2+1$ nodes. The boundary "circle" is a closed loop passing through these points, but intersecting G_{4m+2} in no other points. The graph G_{10} is shown in Figure **??**. The boundary circle is not shown.

(3) Let $n = 4m+3$. The boundary circle for G_{4m+3} is the circle of radius $m+1$, centered at $(0, 0)$. The nodes of G_{4m+3} are the points (h, j) for integers h and j with $0 \le h \le m+1$ and $0 \le j < 4m+3$. The radial edges are the radial line segments joining (h, j) to $(h+1, j)$ for each $0 \le h \le m$ and each $0 \le j < 4m+3$. The circular edges are the circular arcs joining (h, j) to $(h, j+1)$ for each $1 \le h \le m$ and each $0 \le j < 4m+3$. The graph

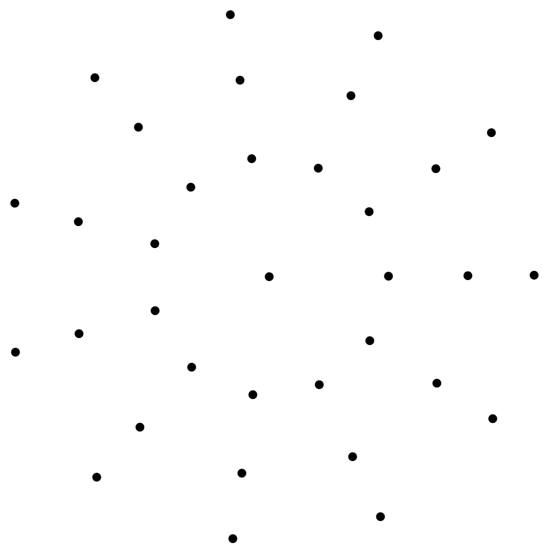


Figure 2-12: Graph G_{11}

G_{4m+3} has $(2m+1)(4m+3)$ edges and $(m+1)(4m+3)+1$ nodes. The graph G_{11} is shown in Figure **??**. The dashed circle is the boundary circle. The circular arcs on the boundary circle are not edges in the graph G_{4m+3}.

(4) Let $n = 4m$. The boundary circle for G_{4m} is the circle of radius m, centered at $(0,0)$. The nodes of G_{4m} are the points (h, j) for integers h and j with $0 \leq h \leq m$ and $1 \leq j \leq 4m$. The radial edges are the radial line segments joining (h, j) to $(h+1, j)$ for each $0 \leq h \leq m-1$ and each $1 \leq j \leq 4m$. The circular edges are the circular arcs joining (h, j) to $(h, j+1)$ $1 \leq h < m$ and also the circular arcs joining (m, j) to $(m, j+1)$ for each *odd* value of j with $0 \leq j < 2m$, and also the circular arcs joining $(m, 4m-j)$ to $(m, 4m-j-1)$ for each *odd* value of j with $0 \leq j < 2m$. There are $(4m-1)(2m)$ edges in G_{4m+1} and there are $4m^2+1$ nodes. The graph G_8 is shown in Figure **??**. The boundary circle is not shown.

Proposition 2.3 *For each integer $n \geq 3$, the graph G_n is well-connected and critical.*

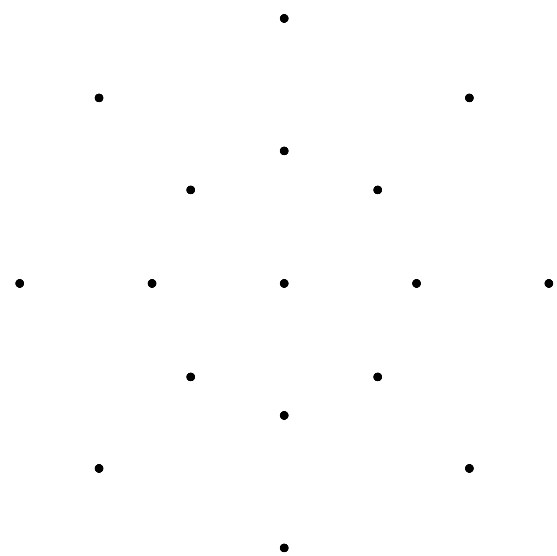


Figure 2-13: Graph G_8

Proof: We will show that G_{4m+1} is well-connected. (The proof that G_{4m+1} is critical is postponed to Chapter **??**.) Suppose $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is a circular pair of boundary points. Assume the indexing of the nodes is such that $p_i = (m+1, p_i)$ and $q_i = (m+1, q_i)$ with

$$0 \leq p_1 < \ldots < p_k < q_k < \ldots < q_1 < 4m+1$$

For each positive integer m, and integers s and t, define paths in G_{4m+1} as follows.

- $\epsilon(m; s, t)$ is the counter-clockwise path along the circle of radius m from (m, s) to (m, t),
- $\delta(m; s, t)$ is the clockwise path along the circle of radius m from (m, s) to (m, t).

For each pair of non-negative integers m, m', and non-negative integer s,

- $\xi(m, m'; s)$ is the path along the ray from (m, s) to (m', s)

The notations $\epsilon(m; s, t)$, $\delta(m, m'; s)$ and $\xi(m; s, t)$ will be abbreviated ϵ, δ and ξ respectively, when the endpoints are clear from the context. If ν is any of the paths ϵ, δ or ξ, the notation $(a,b) \stackrel{\nu}{\to} (c,d)$ means the path ν from the vertex (a,b) to vertex (c,d) along either a circular arc or a radial ray.

The paths in the connection $P \leftrightarrow Q$ are as follows. For $1 \leq j \leq m$, p_j is connected to q_j by the path

$$p_j = (m+1,p_j) \stackrel{\xi}{\to} (m+1-j,p_j) \stackrel{\epsilon}{\to} (m+1-j,q_j) \stackrel{\xi}{\to} (m+1,q_j) = q_j$$

For $m+1 \leq j \leq k$, p_j is connected to q_j by the path

$$p_j = (m+1,p_j) \stackrel{\xi}{\to} (j-m,p_j) \stackrel{\delta}{\to} (j-m,q_j) \stackrel{\xi}{\to} (m+1,q_j) = q_j$$

This gives a k-connection from $P \leftrightarrow Q$ through G_{4m+1}. Since $(P;Q)$ was an arbitrary pair of sequences of boundary nodes in circular order, this shows that G_{4m+1} is well-connected when $n = 4m+1$.

The proof that G_{4m+3} is well-connected is similar; one of the paths goes through the center node $(0,0)$. The proofs for $n = 4m+2$ or $n = 4m$ are also similar but require taking some care with the slight irregularity of the nodes and edges on the outermost circle. The proof that each G_n is critical is postponed to Chapter **??**, where another circular planar graph H_n which is $Y - \Delta$ equivalent to G_n will be described. The proof that H_n is both well-connected and critical is much easier than for the graphs G_n described above. Corollary **??** shows that G_n is also well-connected and critical.

Example 2.4 Figure **??** illustrates a 3-connection $(P \leftrightarrow Q)$, through G_7, where $P = (v_1,v_2,v_3)$ and $Q = (v_4,v_5,v_6)$. In this drawing of G_7, the circular arcs have been replaced by straight lines. The paths in the connection are indicated by solid lines. The edges of the graph G_7 not used in the connection are indicated by dotted lines. This connection is closely related to the paths called *principal flow paths* which will be described in Section **??**.

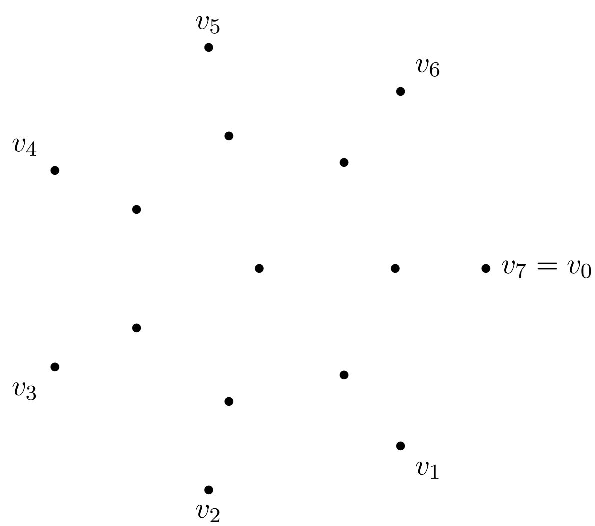


Figure 2-14: 3-connection through G_7

Chapter 3

Resistor Networks

3.1 Conductivities on Graphs

A *conductivity* on a graph G is a function γ which assigns to each edge e in G a positive real number $\gamma(e)$, called the *conductance* of the edge e. A *resistor network* $\Gamma = (G, \gamma)$ is a graph G together with a conductivity function γ.

• "Resistor network" is the standard term for a graph with resistors as edges. The conductance of a resistor is the reciprocal of the resistance. For algebraic reasons, conductance is more convenient than resistance.

If Γ is a resistor network with boundary, the set V_B of *boundary nodes* will sometimes be denoted ∂G, and the set $I = V - V_B$ of *interior nodes* of G will sometimes be denoted int G.

If u is a function defined on all the nodes of a resistor network Γ, and e is an edge of G, with endpoints p and q, the current $c(e)$ through edge e is defined by *Ohm's Law:*

$$c(e) = \gamma(e)[u(p) - u(q)]$$

If there is one or more edges joining p to q in G, $\gamma(p, q)$ is defined to be the sum of the conductances of edges joining p to q. The current from p to q is

$$c(p, q) = \gamma(p, q)[u(p) - u(q)]$$

For each node p in G, the set of nodes q for which there is an edge joining p to q is called the set of *neighbors* of p and is denoted $\mathcal{N}(p)$. A function u

defined on the nodes of G is said to be γ-*harmonic* at p if the (algebraic) sum of the currents from p to the neighboring nodes is 0. That is:

$$\sum_{q \in \mathcal{N}(p)} \gamma(p,q)\,[u(p) - u(q)] = 0 \tag{3.1}$$

If u is γ-harmonic at each of the interior nodes, u is said to be a γ-harmonic function. At a node p where u is not γ-harmonic, *Kirchhoff's Law* says that the current $\phi(p)$ into the network at p must be equal to the (algebraic) sum of the currents from p to its neighboring nodes. That is,

$$\phi(p) = \sum_{q \in \mathcal{N}(p)} \gamma(p,q)\,[u(p) - u(q)]$$

Summing $\phi(p)$ for all nodes p in G, and observing that the current across each edge occurs twice with opposite signs, gives

$$\sum_{p \in G} \phi(p) = 0 \tag{3.2}$$

That is, the (algebraic) sum of the currents into Γ at all nodes is 0. If u is a γ-harmonic function on a resistor network with boundary, then $\phi(p) = 0$ at all interior nodes, and equation **??** becomes

$$\sum_{p \in \partial G} \phi(p) = 0$$

Suppose that $\Gamma = (G, \gamma)$ is a resistor network with n boundary nodes and d interior nodes; there are $m = n + d$ nodes altogether. If f is a function defined on the boundary nodes, there will be a unique function u which agrees with f at the boundary nodes, and is γ-harmonic at the interior nodes of Γ. This function u on Γ is called the *potential* due to f. The response matrix Λ gives the current flow $\phi = \Lambda f$ at the boundary due to the potential u. Several ways to construct Λ will be described, and then used to derive some of the properties of γ-harmonic functions on a resistor network.

(I) Direct use of Kirchhoff's Law. Kirchhoff's Law can be used to establish elementary properties of γ-harmonic functions such as the maximum principle for values and the maximum principle for currents. Kirchhoff's Law can also be used to construct γ-harmonic functions with certain prescribed boundary data, by a process called *harmonic continuation*. In Chapter **??**, harmonic continuation will be used to show the existence of the special functions on networks needed in the algorithm for the recovery of conductances from the response matrix.

(II) THE KIRCHHOFF MATRIX. Suppose $\Gamma = (G, \gamma)$ is a connected resistor network, for which the underlying graph G has a total of m nodes. The Kirchhoff matrix (see Section **??**) is an $m \times m$ matrix K which has the following interpretation. If u is a function, not necessarily γ-harmonic, defined at all the nodes of G, and u is considered to be a voltage, then $\phi = Ku$ is the resulting current flow into Γ. If Γ is a connected resistor network with boundary, the response matrix Λ_γ can be obtained as the Schur complement in K of the square submatrix corresponding to the interior nodes of Γ. This will be used to establish certain properties of Λ_γ, especially the close relation between subdeterminants of Λ_γ and connections through the graph G.

(III) THE DIRICHLET NORM. If $\Gamma = (G, \gamma)$ is a resistor network, there is a quadratic form $W_\gamma(u)$ which is the discrete analogue of the Dirichlet norm for functions defined on a continuous media. For any function u defined at all the nodes of G,

$$W_\gamma(u) = \sum_{pq} \gamma(p,q)\,(u(p) - u(q))^2$$

The sum is taken over all edges pq in G. W_γ has the following minimizing property. If the values of u are fixed at the boundary nodes, then $W_\gamma(u)$ achieves its minimum value when u is γ-harmonic at each interior node. Under certain restrictions, if some boundary values and some boundary currents are specified, the function W_γ can be used to show that there is a unique γ-harmonic function on Γ with this boundary data. See Section **??**.

We begin with (I). If u is a γ-harmonic function on a resistor network $\Gamma = (G, \gamma)$, then for each interior node p, equation **??** can be rewritten as:

$$\left(\sum_{q \in \mathcal{N}(p)} \gamma(p,q) \right) u(p) = \sum_{q \in \mathcal{N}(p)} \gamma(p,q) u(q) \tag{3.3}$$

This says that if u is a γ-harmonic function on Γ, then the value of u at each interior node is the weighted average of the values of u at the neighboring

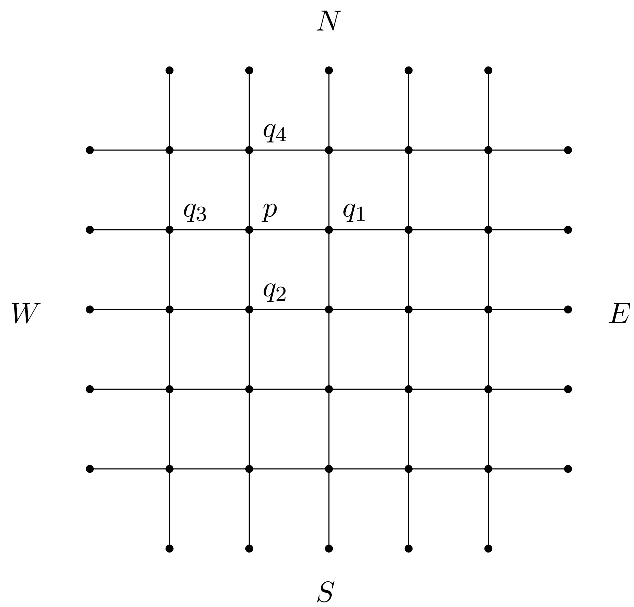


Figure 3-1: Node with four neighboring nodes

nodes. The weights are positive, because γ is assumed to be a positive function on the edges of G. For a rectangular graph, illustrated in the square network of Figure **??**, this is a 5-point formula at each interior node. If any four of the values $u(q_1)$, $u(q_2)$, $u(q_3)$ and $u(q_4)$, $u(p)$, are known, the fifth value is determined by equation **??**. Similarly, if the values of $u(q_2)$, $u(q_3)$ and $u(q_4)$, and the current across the edge q_3p are known, the value of $u(q_1)$ can be found by Kirchhoff's Law.

Throughout the remainder of this section, $\Gamma = (G, \gamma)$ is a resistor network with boundary, and it is assumed that each connected component of the graph G contains at least one boundary point. To say that u is a γ-harmonic function on Γ, means that u is a function defined at all the nodes of G, and u satisfies equation **??** (or equivalently equation **??**) at each interior node. The following is an immediate consequence of the averaging property for γ-harmonic functions.

Lemma 3.1 *Suppose u is a γ-harmonic function on Γ, and let p be an interior node. Then either $u(p) = u(q)$ for all nodes $q \in \mathcal{N}(p)$ or there is at least one node $q \in \mathcal{N}(p)$ for which $u(p) > u(q)$ and there is at least one node $r \in \mathcal{N}(p)$ for which $u(p) < u(r)$.*

If u is a γ-harmonic function on Γ and if the maximum value of u were to occur at an interior node, then the value of u at all the neighbors would be the same. Thus either u is a constant or the maximum value does not occur at an interior node and so must occur at one or more of the boundary nodes. Similarly the minimum value of u must occur on the boundary of Γ. This proves the following.

Theorem 3.2 (Maximum Principle for Harmonic Functions) *Suppose u is a γ-harmonic function on a resistor network Γ with boundary. Then the maximum and minimum values of u occur on the boundary of Γ.*

Corollary 3.3 *If u is a γ-harmonic function on Γ such that $u(p) = 0$ for all boundary nodes p. Then $u(p) = 0$ for all interior nodes.*

There is also a maximum principle for currents.

Theorem 3.4 (Maximum Principle for Currents) *Suppose u is a γ-harmonic function on a resistor network Γ with boundary, Then the current across any edge pq is less than or equal to the sum of the positive currents into the boundary nodes.*

Proof: Assume that $u(p) > u(q)$. A subgraph H of G is constructed in stages as follows. Let H_0 be the node p. Next let H_1 consist of all nodes r and edges rp in G, such that r is a neighbor of p and $u(r) > u(p)$. Inductively, having defined H_j, let H_{j+1} consist of all edges in H_j and all nodes s and edges st in G such that $t \in H_j$ and $u(s) > u(t)$. This gives an increasing sequences of subnetworks

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$$

Eventually no new edges are added and the process ends. Let H be the union of the H_j. By restricting the conductivity function γ to the edges in the subgraph H, (H, γ) may be considered a resistor network. For each node r in H, let $\psi(r)$ be the (algebraic) sum of the currents from r to its neighboring nodes in H, and let $\phi(r)$ be the (algebraic) sum of the currents from r to its neighboring nodes in G. By the construction of H, $\psi(r) \leq \phi(r)$ for all nodes r in H. For all nodes r in H which are interior nodes of G, $\psi(r) \leq 0$, so the function $\psi(r)$ can be positive only at a node of H which is a boundary node of G. Then

$$\begin{aligned} c(p,q) &\leq -\sum_{r\in \mathrm{int}G\cap H} \psi(r) \\ &\leq \sum_{r\in \partial G\cap H} \psi(r) \\ &\leq \sum_{r\in \partial G\cap H} \phi(r) \end{aligned}$$

Thus the current $c(p,q)$ is less than or equal to the sum of the positive currents into Γ at the boundary nodes of G.

3.2 The Response Matrix

Suppose $\Gamma = (G, \gamma)$ is a resistor network which has n boundary nodes, and d interior nodes. The nodes are indexed so that $\{v_1, \ldots, v_n\}$ are the boundary nodes and $\{v_{n+1}, \ldots, v_{n+d}\}$ are the interior nodes. If the values of a function f are specified at the boundary nodes, the extension of f to a function u which is γ-harmonic at all the interior nodes can be obtained as follows. At each interior node, equation **??** is a linear equation for the d unknown quantities which are the values of u at the interior nodes. This gives a linear system of d equations in d unknowns. Anticipating notation to come later, this system can be written as

$$Dg = b$$

The entries of the matrix D are obtained from the values of the conductors. g stands for the vector of values of u at the d interior nodes, and the values of b are obtained by moving, to the right hand side, in each of the equations expressing Kirchhoff's Law, each term involving $u(v_i)$ where v_i is a boundary node. The values in the vector b will be 0 if and only if $f(v_i) = 0$ for all boundary nodes. The Maximum Principle (**??**) implies that, if $f(v_i) = 0$ at all boundary nodes, then the values of $u(p)$ must be 0 at all interior nodes also. Thus the matrix D is non-singular. Hence, for any assignment of values $f(v_i)$ at the boundary nodes, there is a unique solution g to the matrix equation $Dg = b$. Let u be the vector $u = [f, g]$, where f stands for the first n components of u, namely the given values of f at the boundary nodes of G, and g stands for the last d components of u, which are the values of u at the interior nodes of G obtained by solving the linear system $Dg = b$.

NOTE: Vectors such as u, f, g or b will usually be written as a row vectors, but when they appear in a matrix equation such as $Dg = b$, they are to be interpreted as column vectors. This is to avoid having to write equations such as $Dg^T = b^T$.

This function u is defined on all the nodes of G and is γ-harmonic at the interior nodes. u defines a current ϕ where $\phi(p)$ is the current into the network at boundary node p. Specifically, for each boundary node p,

$$\phi(p) = \sum_{q \in \mathcal{N}(p)} \gamma(p,q)[u(p) - u(q)]$$

The linear map which sends f to ϕ is called the *network response*. If the standard basis is used to represent this linear map, the resulting $n \times n$ matrix is called the *response matrix*, and is denoted Λ, or Λ_γ if it is necessary to emphasize the dependence on the conductivity function γ. The response matrix is more conveniently defined, and its properties more easily obtained, from the Kirchhoff matrix which is taken up next.

3.3 The Kirchhoff Matrix

Suppose $\Gamma = (G, \gamma)$ is a resistor network, with a total of m nodes $v_1, \ldots, v_m$. The notation

- $\gamma_{i,j}$ stands for the sum of the conductances $\gamma(e)$ taken over all edges e joining v_i to v_j and $\gamma_{i,j} = 0$ if there is no edge joining v_i to v_j.

This function $\gamma_{i,j}$ is a function of pairs of indices which is symmetric in i and j, and $\gamma_{i,j} > 0$ if and only if there is an edge from v_i to v_j in Γ.

The *Kirchhoff matrix* K, which depends on the network Γ and the conductivity function γ, is the $m \times m$ matrix defined as follows.

(1) If $i \neq j$ then $K_{i,j} = -\gamma_{i,j}$

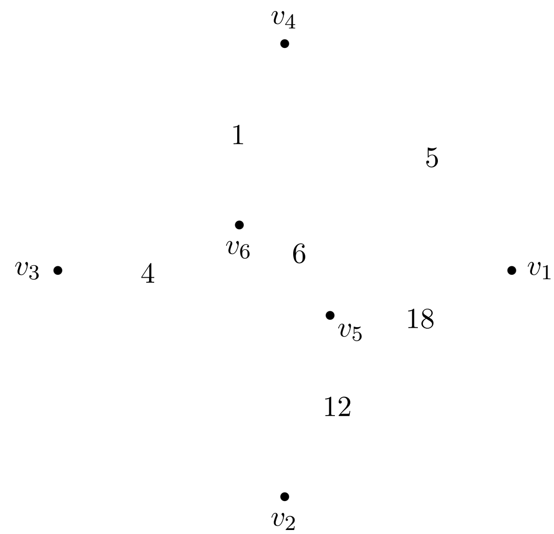


Figure 3-2: Graph G

(2) $K_{i,i} = \sum_{j \neq i} \gamma_{i,j}$

Example 3.1 Let $\Gamma = (G, \gamma)$ be a resistor network whose underlying graph G is shown Figure **??**, and with conductances indicated next to the edges. The Kirchhoff matrix K for this network is:

$$K = \begin{bmatrix} 23 & 0 & 0 & -5 & -18 & 0 \\ 0 & 12 & 0 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 & -4 \\ -5 & 0 & 0 & 6 & 0 & -1 \\ -18 & -12 & 0 & 0 & 36 & -6 \\ 0 & 0 & -4 & -1 & -6 & 11 \end{bmatrix}$$

The Kirchhoff matrix has the following interpretation. If the function u is considered as a voltage defined at the nodes of G, then $\phi = Ku$ is the resulting current (into the network). If a voltage is placed at all the nodes of G, which has value 1 at node j and 0 at all other nodes, then $K_{i,j}$ is the current into Γ at node i. Thus column j of K gives the values of the currents into Γ at nodes $i = 1, \ldots, m$. If u is a voltage whose components are $u = \{u_j\} = \{u(v_j)\}$, then $\phi(v_i) = \sum_j K_{i,j} u_j$ is the current flowing into the network at node v_i. In Section **??** the response matrix will be obtained as a Schur complement of a submatrix in K.

3.4 The Dirichlet Norm

Let $\Gamma = (G, \gamma)$ be a resistor network with a total of m nodes. If x and y are functions defined at the nodes of G, for $i = 1, \ldots, m$, let $x_i = x(v_i)$ and $y_i = y(v_i)$. By a slight abuse of notation, x and y will also stand for the vectors $x = [x_1, \ldots, x_m]$, and $y = [y_1, \ldots, y_m]$ respectively. There is a bilinear form $F_\gamma(x, y)$ obtained from the Kirchhoff matrix by

$$F_\gamma(x, y) = y \cdot K \cdot x^T$$

If this matrix product is expanded out using the definition of the Kirchhoff matrix K, after some algebraic manipulations, the result is:

$$F_\gamma(x, y) = \sum \gamma_{i,j}(x_i - x_j)(y_i - y_j)$$

where the sum is taken over all edges $e = v_i v_j$ in E. Since $\gamma_{i,j}$ is defined to be 0 when there is no edge joining v_i to v_j, this sum can be taken for all pairs i and j with $i < j$. Thus the bilinear form becomes:

$$\begin{aligned}
F_\gamma(x, y) &= \sum_{i<j} \gamma_{i,j}(x_i - x_j)(y_i - y_j) \\
&= \sum_{i<j} \gamma_{i,j}(x_i - x_j)y_i - \sum_{i<j} \gamma_{i,j}(x_i - x_j)y_j \\
&= \sum_{i<j} \gamma_{i,j}(x_i - x_j)y_i + \sum_{i>j} \gamma_{i,j}(x_i - x_j)y_i \\
&= \sum_i y_i \left(\sum_j \gamma_{i,j}(x_i - x_j) \right)
\end{aligned}$$

Here the symmetry of $\gamma_{i,j}$ is used to rewrite the second summand in line two to give line three, and then the terms in line three are combined to give line four. Since $\gamma_{i,j}$ is symmetric and satisfies $\gamma_{i,i} = 0$, this bilinear form can be expressed in the more usual way as a sum over *all* pairs of indices i, j as

$$F_\gamma(x, y) = \frac{1}{2} \sum_{i,j} \gamma_{i,j}(x_i - x_j)(y_i - y_j)$$

The quadratic form associated to the bilinear form $F_\gamma(x, y)$ is denoted W_γ. If $x = [x_1, \ldots, x_m]$, is any function defined on the nodes of Γ, then

$$xKx^T = W_\gamma(x) = \frac{1}{2} \sum_{i,j} \gamma_{i,j}(x_i - x_j)^2$$

W_γ is the discrete analogue, for functions x on a network Γ, of the Dirichlet norm for functions defined on a region in the plane which has an underlying conductivity function.

Suppose $\Gamma = (G, \gamma)$ is a connected resistor network with boundary. Let x be a function defined at the nodes of G. The current into the network at node v_i will be denoted $\phi(v_i)$, or $\phi_x(v_i)$ if it is necessary to specify the dependence on the function x. Ohm's Law says that the current from node v_i to node v_j is $\gamma_{i,j}(x_i - x_j)$. Kirchhoff's Law says that the current into the network at node v_i must be the same as the sum of the currents from node v_i to its neighbors. That is, for each node v_i

$$\phi_x(v_i) = \sum_{v_j \in \mathcal{N}(v_i)} \gamma_{i,j}(x_i - x_j)$$

If the function x is γ-harmonic at each interior node v_i of Γ, then $\phi_x(v_i) = 0$, and the bilinear form becomes:

$$\begin{aligned} F_\gamma(x, y) &= \sum_i y_i \left(\sum_j \gamma_{i,j}(x_i - x_j) \right) \\ &= \sum_i y(v_i) \cdot \phi_x(v_i) \\ &= \sum_{i \in \partial G} y(v_i) \cdot \phi_x(v_i) \end{aligned}$$

The last sum is taken over only the boundary nodes v_i in Γ. This is a discrete analogue of one of Green's identities for the Dirichlet form $F_\gamma(x, y)$.

Similarly,

$$F_\gamma(x, y) = \sum_{i \in \partial G} \phi_y(v_i) \cdot x(v_i)$$

The equality

$$\sum_{i \in \partial G} y(v_i) \cdot \phi_x(v_i) = F_\gamma(x, y) = \sum_{i \in \partial G} \phi_y(v_i) \cdot x(v_i)$$

is the discrete analogue of another of Green's identities.

Lemma 3.5 *Suppose u is a function on Γ which is γ-harmonic at all interior nodes, and let y be any function with $u(p) = y(p)$ for all boundary nodes p. Then*

$$W_\gamma(u) \leq W_\gamma(y)$$

Proof: Write $y = u + z$, where $z = 0$ on ∂G. Then, using Green's identity,

$$\begin{aligned} F_\gamma(u, y) &= \sum_p z(p)\phi_u(p) \\ &= \sum_{p \in \partial G} z(p)\phi_u(p) + \sum_{p \in \mathrm{int} G} z(p)\phi_u(p) \end{aligned}$$

The first term is 0 since $z = 0$ on ∂G, and the second term is 0 since $\phi_u(p) = 0$ at all nodes p in the interior of G. Thus

$$\begin{aligned} W_\gamma(y) &= W_\gamma(u) + 2F_\gamma(u, z) + W_\gamma(z) \\ &= W_\gamma(u) + W_\gamma(z) \\ &\geq W_\gamma(u) \end{aligned}$$

Lemma 3.6 *Suppose $W_\gamma(u) \leq W_\gamma(y)$ for all functions y on G with $y = u$ on ∂G. Then $\phi_u(p) = 0$ for all $p \in$ int G. Thus u is γ-harmonic on G.*

Proof: Suppose z is a fixed function on G, with $z = 0$ on ∂G. Let $y_t = u + tz$, where t is a real parameter and let $f(t) = W_\gamma(y_t)$. This function $f(t)$ has a minimum at $t = 0$. Expanding,

$$f(t) = W_\gamma(u) + 2tF_\gamma(u, z) + t^2 W_\gamma(z)$$

Thus $f'(0) = 2F_\gamma(u, z) = 0$, which shows that any function z defined on the nodes of G which is 0 on ∂G is orthogonal to u. This implies that $\sum z(p)\phi_u(p) = 0$. For each node $p \in \text{int } G$, let z be the function which is 1 at p and 0 for all other nodes. Then $\phi_u(p) = 0$. Thus u is γ-harmonic.

This leads to a quite general set of conditions on boundary values and boundary currents that may be imposed, and for which there is a unique potential. Suppose Γ is a connected resistor network where V is the set of nodes, ∂G a non-empty connected subset of V designated as the set of boundary nodes, and $I = V - \partial G$ is the set of interior nodes. For each boundary node v_i either (but not both) of the following conditions may be imposed:

(1) The value of the function $x(v_i)$.

(2) The value of the current $\phi(v_i)$.

The value of x must be specified at least one boundary node. Suppose there are a total of m nodes of G which are numbered so that $v_1, \ldots, v_n$ are the boundary nodes and $v_{n+1}, \ldots, v_m$ are the interior nodes. Suppose that $1 \le n_1 \le n$ and that the values of x are specified for the nodes v_i for $1 \le i \le n_1$; the values of the current are specified for $n_1 < i \le n$; the function is to be γ-harmonic for $n < i \le m$. That is, suppose values $\{a_i\}$, and $\{b_i\}$ are given, and it is required that:

(1) $x_i = a_i$ for $1 \le i \le n_1$

(2) $\phi(v_i) = b_i$ for $n_1 < i \le n$

(3) $\phi(v_i) = 0$ for $n < i \le m$

The equations in (1), (2) and (3) are a total of m linear equations for the m values $x_1, \ldots, x_m$. (Some of these equations give the values x_i directly, namely the first n_1 equations.) To show that matrix for this system is non-singular, suppose there were two solutions $x = \{x_i\}$ and $y = \{y_i\}$ with the same boundary conditions. For each $1 \le i \le m$, let $z_i = x_i - y_i$. Then $z = \{z_i\}$ would be a solution to the system

(1) $z_i = 0$ for $1 \le i \le n_1$

(2) $\sum_j \gamma_{i,j}(z_i - z_j) = 0$, for $n_1 < i \le m$

Since the function z is γ-harmonic at all nodes v_i for $n_1 < i \leq m$, z must be the function for which

$$W_\gamma(z) = \sum_{i<j} \gamma_{i,j}(z_i - z_j)^2$$

achieves its minimum value. Since $z_i = 0$ for $1 \leq i \leq n_1$, this minimum value can be 0, and so must be 0. Hence $z_i = z_j$ for each of the neighbors of z_i. Since G is assumed to be connected as a graph, the values of z must be constant, and the constant must be 0, since there is at least one boundary node, and thus at least one equation $z_i = 0$. Therefore the system always has a unique solution.

The assumption that the graph G be connected is not strictly necessary. It need only be assumed that in each connected component of G there is at least one boundary node at which the value of x is specified. With this proviso, boundary conditions can be specified by imposing either a value for the function or a value for the current at each boundary node, and there will be a unique potential on G with this boundary data.

The bilinear form $F_\gamma(x, y)$ is related to the bilinear form $< g, \Lambda_\gamma f >$ defined by the response matrix Λ_γ as follows. Suppose f and g are functions defined on the boundary nodes. Let x and y be the potentials due to f and g respectively. For each boundary node v_i, let $\phi_f(v_i)$ be the boundary current due to the function f. Then

$$\begin{aligned} F_\gamma(x, y) &= \sum g(v_i)\phi_f(v_i) \\ &= < g, \Lambda_\gamma f > \\ &= < f, \Lambda_\gamma g > \end{aligned}$$

There is a quadratic form $Q_\gamma(\cdot)$ associated to the bilinear form $< g, \Lambda_\gamma f >$, as follows. For each function f defined on the boundary nodes of Γ,

$$Q_\gamma(f) = < f, \Lambda_\gamma f >$$

The polarization identity for Q_γ is:

$$Q_\gamma(f + g) - Q_\gamma(f - g) = 4 < g, \Lambda_\gamma f >$$

This shows that the quadratic form Q_γ determines the matrix Λ_γ, and conversely. This quadratic form Q_γ will be used Section **??** and in Chapter **??**.

3.5 The Schur Complement

Suppose $\Gamma = (G, \gamma)$ is a connected resistor network with boundary. V is the set of nodes, V_B is the subset of V designated as boundary nodes, and $I = V - V_B$ is the set of interior nodes.

- $K(I; I)$ is the submatrix of K with index set $= I$.

The response matrix can be obtained by using the sub-matrix $K(I; I)$ to block reduce K. This is best explained by means of the *Schur complement*: $\Lambda = K/K(I; I)$. To make this treatment as self-contained as possible, a brief description of the Schur complement will be given.

Suppose M is a square matrix and D is a non-singular square sub-matrix of M. For convenience, assume that D is the lower right hand corner of M, so that M has the following block structure:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.4}$$

The *Schur complement* of D in M is defined to be the matrix

$$M/D = A - BD^{-1}C \tag{3.5}$$

M/D is the submatrix in the upper left corner that results from using D to block reduce M. The Schur complement satisfies the following determinantal identity:

$$\det M = \det(M/D) \cdot \det D \tag{3.6}$$

If E is a non-singular square sub-matrix of D, then

$$\det M = \det(M/D) \cdot \det(D/E) \cdot \det E$$

In this situation, the quotient formula of [**?**] is:

$$M/D = (M/E)/(D/E) \tag{3.7}$$

Let $A = \{a_{i,j}\}$ be an $n \times n$ matrix, and assume that $a_{h,k}$ is a non-zero entry. The 1×1 matrix with entry $a_{h,k}$ is denoted $[a_{h,k}]$. In this case, the Schur complement, of $[a_{h,k}]$ can be obtained by row-reduction using the single entry $a_{h,k}$. The determinantal identity becomes:

$$\det A = (-1)^{h+k} a_{h,k} \cdot \det(A/[a_{h,k}])$$

At the other extreme, suppose D is an $(n-1) \times (n-1)$ non-singular submatrix of an $n \times n$ matrix A. In this case A/D is the 1×1 matrix consisting of the single entry

$$a' = a_{1,1} - BD^{-1}C$$

The determinantal identity becomes

$$\det A = a' \cdot \det D \tag{3.8}$$

Suppose A is an $n \times n$ matrix, with $n \geq 2$. If i and j are any two indices, $A[i;j]$ will denote the $(n-1) \times (n-1)$ matrix obtained by *deleting* row i and column j. Similarly, if (h,i) and (j,k) are two pairs of indices, then $A[h,i;j,k]$ will denote the $(n-2) \times (n-2)$ matrix obtained by *deleting* rows h and i and *deleting* columns j and k.

We make extensive use of the following identity, due to Sylvester. It was used by Dodgson in [**?**]. It will be called the six-term identity.

Lemma 3.7 THE SIX-TERM IDENTITY *Let A be an $n \times n$ matrix. Then for any indices $[h,i;j,k]$ with $1 \leq h < i \leq n$ and $1 \leq j < k \leq n$,*

$$\det A \cdot \det A[h,i;j,k] = \det A[h;j] \cdot \det A[i;k] - \det A[h;k] \cdot \det A[i;j]$$

Proof: By re-ordering the rows and columns, the indices may be assumed to be $(h,i) = (1,2)$ and $(j,k) = (1,2)$. Let $D = A[1,2;1,2]$. Then A has the form:

$$A = \begin{bmatrix} a & b & x \\ c & d & y \\ w & z & D \end{bmatrix}$$

where x and y are $1 \times (n-2)$ row vectors, w and z are $(n-2) \times 1$ column vectors. Temporarily assume that D is non-singular. The formula for the Schur complement A/D gives:

$$A/D = \begin{bmatrix} a - xD^{-1}w & b - xD^{-1}z \\ c - yD^{-1}w & d - yD^{-1}z \end{bmatrix}$$

$$\begin{aligned} &\det(A/D) = (a - xD^{-1}w)(d - yD^{-1}z) - (b - xD^{-1}z)(c - yD^{-1}w) \\ &= \det(A[2;2]/D) \cdot \det(A[1;1]/D) - \det(A[1;2]/D) \cdot \det(A[2;1]/D) \end{aligned}$$

The determinantal identity for Schur complements gives:

$$\begin{aligned}\det A \cdot \det D &= \det(A/D) \cdot (\det D)^2 \\ &= \det A[2;2] \cdot \det A[1;1] - \det A[1;2] \cdot \det A[2;1]\end{aligned}$$

This is a polynomial relation which holds for the n^2 values of the entries of A whenever $\det D \neq 0$. Therefore it is an identity in the coefficients of A.

Lemma 3.8 *Suppose $\Gamma = (G, \gamma)$ is a connected resistor network. Then the Kirchhoff matrix K is positive semi-definite. If $P = (p_1, \ldots, p_k)$ is a non-empty proper subset of the nodes of G, then the matrix $K(P;P)$ is positive definite.*

Proof: Suppose there are a total of m nodes numbered $v_1, \ldots, v_m$. Let $x = [x_1, \ldots, x_m]$ is a vector of length m. The expression for xKx^T as the Dirichlet norm in equation **??** is

$$xKx^T = \frac{1}{2} \sum_{i,j} \gamma_{i,j}(x_i - x_j)^2 \tag{3.9}$$

which shows that K is positive semi-definite. Suppose $xKx^T = 0$. If $\gamma_{i,j} > 0$, then $x_i = x_j$. Since the graph G is connected, this means that $x_i = x_j$ for all nodes v_i and v_j. Let $A = K(P;P)$, and suppose $y = [y_1, \ldots, y_k]$ is a vector with $yAy^T = 0$. Let $x = [x_1, \ldots, x_m]$ be the vector with $x_{p_i} = y_i$ for $1 \leq i \leq k$, and $x_j = 0$ if j is not in P. Then $xKx^T = yAy^T = 0$. Since P is a proper subset of the vertex set for G, at least one of the x_i is 0. Since G is connected, all the x_i must be 0. Hence the y_i are also 0.

Until now in this section, $\Gamma = (G, \gamma)$ has been assumed only to be a resistor network. For the remainder of the section, $\Gamma = (G, \gamma)$ is a resistor network *with boundary.* The nodes are indexed so that $V_B = \{v_1, \ldots, v_n\}$ is the set of boundary nodes, and $I = \{v_{n+1}, \ldots, v_{n+d}\}$ is the set of interior nodes.

Theorem 3.9 *Suppose $\Gamma = (G, \gamma)$ is a connected resistor network with boundary. Then the network response matrix Λ_γ is the Schur complement*

$$\Lambda_\gamma = K/K(I;I)$$

Proof: If I is the empty set, $K/K(I;I)$ is defined to be K, and $\Lambda_\gamma = K$. Otherwise, I is non-empty. Let $D = K(I;I)$. Then K has a block structure:

$$K = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \tag{3.10}$$

By Lemma **??**, D is non-singular. Suppose that f is a function imposed at the boundary nodes. Let g be the values of the resulting potential at the interior nodes. f will also stand for the vector $[f_1, \ldots, f_n]$, where $f_i = f(v_i)$ for each $1 \le i \le n$. g will also stand for $[g_{n+1}, \ldots, g_{n+d}]$ where $g_i = g(v_i)$ for each $n+1 \le i \le n+d$. The vector of currents into the boundary nodes is denoted ϕ. Kirchhoff's current law says that the sum of the currents into each interior node is 0. Thus

$$\begin{aligned} Af + Bg &= \phi \\ B^T f + Dg &= 0 \end{aligned} \tag{3.11}$$

When written in block form, equations **??** become

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix}$$

Solving equations **??** first for g, and then for ϕ, we have

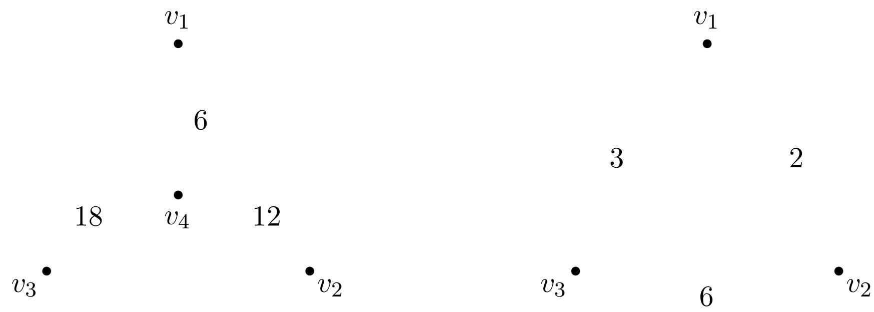
$$\begin{aligned} g &= -D^{-1}B^T f \\ \phi &= (A - BD^{-1}B^T)f \end{aligned}$$

Therefore the response matrix is $\Lambda_\gamma = A - BD^{-1}B^T$, which is the Schur complement of D in K.

Observation 3.1 The values of the potential u at the interior nodes p are:

$$u(p) = g(p) = [-D^{-1}B^T f](p) \tag{3.12}$$

Example 3.2 Suppose that $\Gamma = (G, \gamma)$ is the circular planar graph in the shape of an (inverted) Y, as in Figure **??**a, with three boundary nodes v_1, v_2, and v_3, and one interior node v_4. The conductances of the edges are:



(a) Y (b) $\triangle$

Figure 3-3: Y and $\triangle$

$\gamma(v_1v_4) = 6$, $\gamma(v_2v_4) = 12$ and $\gamma(v_3v_4) = 18$, as indicated on the figure. The Kirchhoff matrix for the network Γ is:

$$K = \begin{bmatrix} 6 & 0 & 0 & -6 \\ 0 & 12 & 0 & -12 \\ 0 & 0 & 18 & -18 \\ -6 & -12 & -18 & 36 \end{bmatrix}$$

In this case, the response matrix Λ is the Schur complement in K of the single entry 6 in the $(4,4)$ position. Using this entry to row-reduce the last column of K, the result is:

$$\Lambda = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 8 & -6 \\ -3 & -6 & 9 \end{bmatrix}$$

The values 2, 3 and 6 are the values of the conductances for the edges v_1v_2, v_1v_3 and v_2v_3 for the graph in the shape of the $\triangle$ in Figure **??**b. Since there are no interior nodes, the off-diagonal entries in the Kirchhoff matrix for the graph $\triangle$ in Figure **??**b are the (negatives of) values of the conductors. The Kirchhoff matrix for the network in Figure **??**b is the same as its response matrix.

If the values of the conductors in a Y (such as that of Figure **??**a) are $\gamma(v_1v_4) = a$, $\gamma(v_2v_4) = b$, $\gamma(v_3v_4) = c$, then the values of the conductors in the $\triangle$ (such as that of Figure **??**b) will be $\gamma(v_1v_2) = \sigma^{-1}ab$, $\gamma(v_2v_3) = \sigma^{-1}bc$, $\gamma(v_1v_3) = \sigma^{-1}ac$, where $\sigma = a+b+c$. A resistor network that contains the Y of Figure **??**a will be electrically equivalent to the resistor network obtained by replacing the conductors of the Y in Figure **??**a by the conductors of the $\triangle$ in Figure **??**b.

Example 3.3 Let $\Gamma = (G, \gamma)$ be the network of Figure **??**, with conductances as indicated adjacent to the edges in the figure. The Kirchhoff matrix K for Γ is:

$$K = \begin{bmatrix} 23 & 0 & 0 & -5 & -18 & 0 \\ 0 & 12 & 0 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 & -4 \\ -5 & 0 & 0 & 6 & 0 & -1 \\ -18 & -12 & 0 & 0 & 36 & -6 \\ 0 & 0 & -4 & -1 & -6 & 11 \end{bmatrix}$$

The response matrix is the Schur complement of the sub-matrix $K(5,6;5,6)$ in K, where

$$K(5,6;5,6) = \begin{bmatrix} 36 & -6 \\ -6 & 11 \end{bmatrix}$$

For this network, the response matrix, calculated as the Schur complement $\Lambda = K/K(5,6;5,6)$, is

$$\Lambda = \begin{bmatrix} 13.1 & -6.6 & -1.2 & -5.3 \\ -6.6 & 7.6 & -0.8 & -0.2 \\ -1.2 & -0.8 & 2.4 & -0.4 \\ -5.3 & -0.2 & -0.4 & 5.9 \end{bmatrix} \tag{3.13}$$

Figure **??** shows a graph H which is not a circular planar graph. H has 6 edges; the conductances for the edges are $\gamma(v_1v_2) = 6.6$, $\gamma(v_2v_3) = 0.8$, $\gamma(v_3v_4) = 0.4$, $\gamma(v_1v_4) = 5.3$, $\gamma(v_1v_3) = 1.2$, $\gamma(v_2v_4) = 0.2$. Since there are no interior nodes, the off-diagonal entries in the response matrix for the graph in Figure **??** are the (negatives of) values of the conductors. Thus the Kirchhoff matrix of the network of Figure **??** is the same as its response matrix which is also the response matrix for the network in Figure **??**.

Suppose the vector of voltages $f = [1,0,0,0]$ is applied to the boundary nodes in the network of Figure **??**. That is the voltage is 1 at node v_1 and

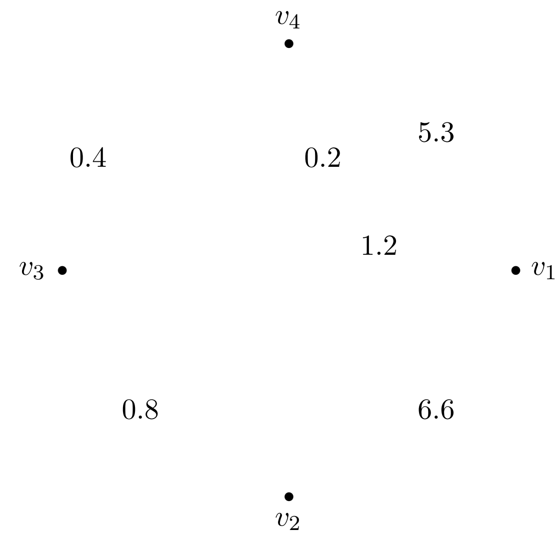


Figure 3-4: Graph H exhibiting the response matrix Λ

0 at nodes v_2, v_3 and v_4. The current into Γ is the first column of Λ, which is:

$$\Lambda \cdot f = [13.1, -6.6, -1.2, -5.3]^T$$

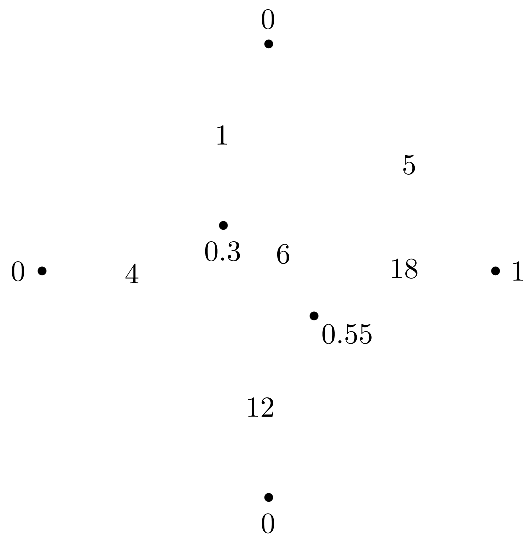
at nodes v_1, v_2, v_3, v_4. The potential at the interior nodes can be found by formula **??**, which says that

$$u(p) = [-D^{-1}B^T f](p)$$

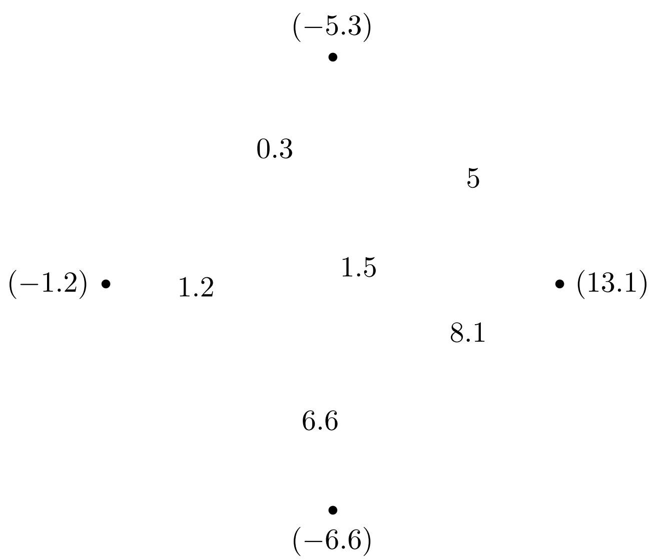
In this case, $u(v_5) = 0.55$ and $u(v_6) = 0.3$. The voltages are indicated in Figure **??**a. The current through each edge and the direction of the current flow is indicated in Figure **??**b. The current into the network at each boundary node is indicated in parentheses; the negative sign means that the current flows out.

3.6 Sub-matrices of the Response Matrix

If $A = (a_1, \ldots, a_s)$ and $B = (b_1, \ldots, b_t)$ are two sequences of nodes, the notation $A + B$ denotes the sequence $(a_1, \ldots, a_s, b_1, \ldots, b_t)$. With this no-



(a) Voltages and conductances



(b) Currents

Figure 3-5: Voltages, Conductances; Currents tation, the following is an immediate consequence of Theorem **??** and the definition of Schur complement.

Lemma 3.10 *Suppose Γ is a connected resistor network with boundary, and let Λ be its response matrix. Let P and Q be two sequences of boundary nodes of Γ. Then the sub-matrix $\Lambda(P;Q)$ of Λ is the Schur complement*

$$\Lambda(P;Q) = K(P+I;Q+I)/K(I;I)$$

Suppose $\Gamma = (G,\gamma)$ is a connected resistor network with boundary, and p is one of the boundary nodes. Let Γ' be the resistor network with the same graph G, with the same conductivity function γ, but p is declared to be an interior node. Let Λ' denote the response matrix for Γ'. By Theorem **??** and the quotient formula **??**,

$$\Lambda' = K/K(I+p;I+p)$$

Suppose $P = (p_1,\ldots,p_k)$ and $Q = (q_1,\ldots,q_k)$ are two sequences of boundary nodes, and p is another boundary node of Γ, with p not in the set $P \cup Q$. The identification of the response matrix as a Schur complement (Theorem **??**), the quotient formula **??**, and the determinantal identity, equation **??**, for Schur complements shows the following.

Lemma 3.11 (1) $\quad \Lambda'(P;Q) = \Lambda(P+p;Q+p)/\Lambda(p;p)$

(2) $\quad \det\Lambda'(P;Q) = \det\Lambda(P+p;Q+p)/\det\Lambda(p;p)$

3.7 Connections and Determinants

Suppose $\Gamma = (G,\gamma)$ is a connected resistor network with boundary. E is the edge-set, V is the vertex set, V_B is the subset of V designated as boundary nodes, and $I = V - V_B$ is the set of interior nodes. The definition of a connection through G will be given in slightly greater generality than in Chapter **??**. If p and q are two boundary nodes, a path from p to q through G is a sequence of edges $e_0 = pr_1, \quad e_0 = r_1r_2, \quad \ldots e_{h-1} = r_{h-1}r_h, \quad e_h = r_hq$ such that $r_1, r_2, \ldots, r_h$ are distinct interior nodes of G. Suppose $P = (p_1,\ldots,p_k)$ and $Q = (q_1,\ldots,q_k)$ are two disjoint sets of boundary nodes. A k-connection $P \leftrightarrow Q$ from P to Q through G is a set $\alpha = (\alpha_1,\ldots,\alpha_k)$ of disjoint paths through G, where for each $1 \leq i \leq k$, α_i is a path from P_i to

$Q_{\tau(i)}$, and τ is an element of the permutation group S_k. This is more general than the previous definition of k-connection, because here the graph G is not assumed to be planar. There may be two (or more) k-connections α and β from P to Q where the endpoints of α and β are different permutations of the nodes in Q. This cannot occur if $(P; Q)$ are a circular pair of boundary nodes in a circular planar graph.

Let $\mathcal{C}(P; Q)$ be the set of connections from P to Q. For each k-connection $\alpha = (\alpha_1, \dots, \alpha_k)$ in $\mathcal{C}(P; Q)$, let

- τ_α be the permutation of the nodes $(q_1, q_2, \dots, q_k)$ which results at the final endpoints of the paths $(\alpha_1, \alpha_2, \dots, \alpha_k)$;
- E_α be the set of edges in α;
- J_α be the set of interior nodes which are not the endpoints of any edge in α.
- $D_\alpha = \det K(J_\alpha; J_\alpha)$

Lemma 3.12 *Suppose $\Gamma = (G, \gamma)$ is a connected resistor network with boundary. Let $P = (p_1, p_2, \dots, p_k)$ and $Q = (q_1, q_2, \dots, q_k)$ be two disjoint sequences of boundary nodes. Then*

$$\det \Lambda(P; Q) \cdot \det K(I; I) = (-1)^k \sum_{\tau \in S_k} \mathrm{sgn}(\tau) \left\{ \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \prod_{e \in E_\alpha} \gamma(e) \cdot D_\alpha \right\}$$

Proof: Suppose there are n boundary nodes, d interior nodes, and $m = n+d$ nodes altogether. Let $\nu = k+d$. Let the interior nodes be numbered r_i for $i = k+1, \dots, k+d$. (This is a convenient re-indexing of the interior nodes $v_{n+1}, \dots, v_{n+d}$.) Taking the Schur complement of $K(I; I)$, the determinantal formula **??** gives

$$\det \Lambda(P; Q) \cdot \det K(I; I) = \det K(P + I; Q + I)$$

Temporarily let the $\nu \times \nu$ matrix $K(P+I;Q+I)$ be denoted by $M = \{m_{i,j}\}$. Then

$$\det M = \sum_{\sigma \in S_\nu} \mathrm{sgn}(\sigma) \prod_{i=1}^{\nu} m_{i,\sigma(i)}$$

Here S_ν denotes the symmetric group on ν symbols. Fix $\sigma \in S_\nu$, and for each $1 \leq i \leq k$, let n_i be the first index j for which $\sigma^j(i) \leq k$. For each $1 \leq i \leq k$, and $0 \leq j \leq n_i$, let $a(i,j) = \sigma^j(i)$. Let τ be the permutation of $1, 2, \ldots, k$ where $\tau(i) = a(i, n_i)$. Thus each $\sigma \in S_\nu$ gives a diagram of the following form:

$$1 = a(1,0) \overset{\sigma}{\mapsto} a(1,1) \overset{\sigma}{\mapsto} a(1,2) \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} a(1,n_1) = \tau(1)$$

$$2 = a(2,0) \overset{\sigma}{\mapsto} a(2,1) \overset{\sigma}{\mapsto} a(2,2) \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} a(2,n_2) = \tau(2)$$

$$\ldots$$

$$k = a(k,0) \overset{\sigma}{\mapsto} a(k,1) \overset{\sigma}{\mapsto} a(k,2) \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} a(k,n_k) = \tau(k)$$

Let A be the subset of $\{1, 2, \ldots, \nu\}$ consisting of the $a(i,j)$ for $1 \leq i \leq k$ $0 \leq j < n_i$. Let $t = \sum n_i$, which is the cardinality of A. Let B be the set $\{1, 2, \ldots, \nu\} - A$. Then σ may be expressed as a product $\sigma = \phi \cdot \mu$, where ϕ is a permutation of A, and μ is a permutation of B. Let ϕ be expressed as a product of disjoint cycles $\phi = \phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_s$. Then $\mathrm{sgn}(\sigma) = (-1)^{t-s}\mathrm{sgn}(\mu)$. Then τ will also be expressed as a product of s cycles. $\tau = \psi_1 \cdot \psi_2 \cdots \psi_s$ and $\mathrm{sgn}(\tau) = (-1)^{k-s}$. Thus $\mathrm{sgn}(\sigma) = (-1)^{k+t}\mathrm{sgn}(\tau)\mathrm{sgn}(\mu)$.

The diagram above determines a set $\alpha = (\alpha_1, \ldots, \alpha_k)$ of sequences of nodes in G, where α_i is the sequence $a(i,0), a(i,1), \ldots, a(i,n_i)$. For each $1 \leq i \leq k$, $a(i,0) = p_i$ and $a(i,n_i) = q_{\tau(i)}$. For each $1 \leq i \leq k$, and $0 < j < n_i$, $a(i,j)$ is the interior node $r_{a(i,j)}$. The product $\prod_{i=1}^{\nu} m_{i,\sigma(i)}$ can be non-zero only if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ forms a connection through G from P to Q. For each $\alpha \in \mathcal{C}(P;Q)$, let $S(\alpha)$ be the set of $\sigma \in S_\nu$ for which the connection is α. As σ varies over $S(\alpha)$, μ varies over the permutations of

J_α. Then

$$\begin{aligned}
&\sum_{\sigma\in S(\alpha)} \operatorname{sgn}(\sigma)\prod_{i=1}^{\nu} m_{i,\sigma(i)} \\
&\qquad = \sum_{\sigma\in S(\alpha)} (-1)^{k+t}\operatorname{sgn}(\tau)\prod_{e\in E_\alpha}(-\gamma(e))\cdot\operatorname{sgn}(\mu)\cdot\prod_{i\in J_\alpha} m_{i,\mu(i)} \\
&\qquad\qquad = (-1)^{k}\operatorname{sgn}(\tau)\cdot\prod_{e\in E_\alpha}\gamma(e)\cdot\det K(J_\alpha;J_\alpha).
\end{aligned}$$

For each $\tau \in S_k$, take the sum over all α which induce this τ. Then take the sum over all $\tau \in S_k$, and the proof is complete.

Observation 3.2 Comments on the proof of Lemma **??**.

(1) $K(P+I;Q+I)$ is a $\nu \times \nu$ matrix. Every $\sigma \in S_\nu$ corresponds to a potential path α from P to Q through G, which might be a connection from $(p_1,\ldots,p_k)$ to $(q_{\tau(1)},\ldots,q_{\tau(k)})$. The path α_i is the sequence

$$p_i = a(i,0) \overset{\sigma}{\mapsto} a(i,1) \overset{\sigma}{\mapsto} a(i,2) \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} a(i,n_i) = q_{\tau(i)}$$

where each $a(i,j)a(i,j+1)$ might be an edge in G.

(2) If for each $1 \le i \le k$, and each $0 \le j \le n_i - 1$, the value of $\gamma(a(i,j)a(i,j+1))$ is non-zero, then the diagram corresponds to an actual path from $(p_1,\ldots,p_k)$ to $(q_{\tau(1)},\ldots,q_{\tau(k)})$.

(3) α is an actual path if and only if $\prod_{i=1}^{\nu} m_{i,\sigma(i)} \neq 0$.

(4) The analysis of the signs of the permutations shows that for each fixed $\tau \in S_k$, all the signs of the terms in the expansion of the determinant are all the same, namely $(-1)^k\operatorname{sgn}(\tau)$.

Observation 3.3 A consequence of Lemma **??** is that if $\det\Lambda(P;Q) = 0$, then one or the other of the following two possibilities is true.

(1) There is no connection from P to Q.

(2) There are (at least) two connections α and β from P to Q, with permutations τ_α and τ_β of opposite sign.

Theorem 3.13 *Suppose* $\Gamma = (G, \gamma)$ *is a circular planar resistor network and* $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ *is a circular pair of sequences of boundary nodes.*

(a) If $(P; Q)$ *are not connected through* G*, then* $\det \Lambda(P; Q) = 0$.

(b) If $(P; Q)$ *are connected through* G*, then* $(-1)^k \det \Lambda(P; Q) > 0$.

Proof: Part (a) follows directly from **??**. For part (b), first consider the case when G is connected as a graph. By Lemma **??**, $K(I; I)$ is positive definite, so $\det K(J; J) > 0$ for all $J \subseteq I$. The sequence $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order around the boundary of G. If there is a connection from P to Q, it must connect p_i to q_i for $1 \leq i \leq k$. Thus each permutation τ which appears in Lemma **??** is the identity, so all the terms in the sum have the same sign. In the general case, G is a disjoint union of connected components G_i, and $\Lambda = \Lambda(\Gamma, \gamma)$ is a direct sum of the $\Lambda(\Gamma_i, \gamma_i)$.

Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are two sequences of boundary nodes that are in disjoint arcs of the boundary circle. Let P' and Q' be the permutations of P and Q respectively, so that $(P'; Q')$ are a circular pair. Then

$$\det \Lambda(P; Q) = \pm \Lambda(P'; Q')$$

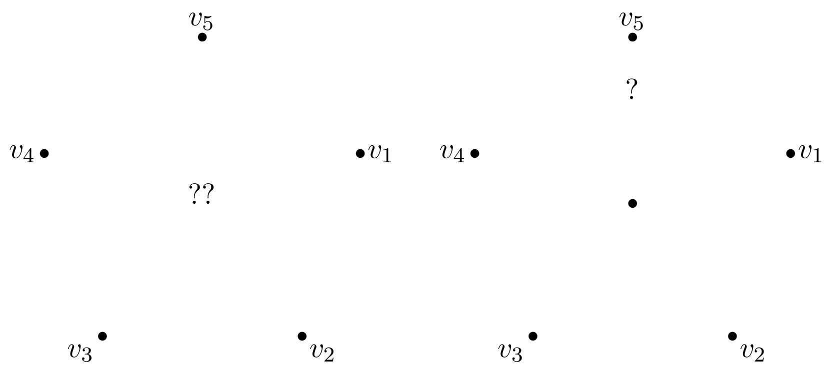
Corollary 3.14 *Suppose* $\Gamma = (G, \gamma)$ *is a circular planar resistor network with* P *and* Q *are arbitrary sequences (of the same length) of boundary nodes which are in disjoint arcs of the boundary circle. Then* $\Lambda(P; Q)$ *is non-singular if and only if* P *and* Q *are connected through* G*. In particular, if* G *is well-connected as a graph, then* $\Lambda(P; Q)$ *is non-singular.*

This Corollary of Theorem **??** is useful for showing the existence of solutions to linear systems of the form $\Lambda(P; Q)x = c$, where P and Q are sets of boundary nodes in disjoint arcs of the boundary circle, but P and Q are not necessarily in circular order.

Example 3.4 Let $\Gamma = (G, \gamma)$ be a resistor network with five boundary nodes as in Figure **??**a. Suppose it is known that

(1) there is a 1-connection from v_1 to v_4

(2) there is a 1-connection from v_2 to v_3



(a) Paths unknown (b) Deduced Connections

Figure 3-6: Example **??**

(3) there is no 2-connection from $P = (v_1, v_2)$ to $Q = (v_4, v_3)$.

Statement (3) and Theorem **??** imply that

$$\det \Lambda(P; Q) = \det \begin{bmatrix} \Lambda(1,4) & \Lambda(1,3) \\ \Lambda(2,4) & \Lambda(2,3) \end{bmatrix} = 0$$

The diagonal entries are both non-zero, so the off-diagonal entries must both be non-zero also. Thus there must be:

(4) a 1-connection from v_1 to v_3, and also

(5) a 1-connection from v_2 to v_4.

Then there must also be

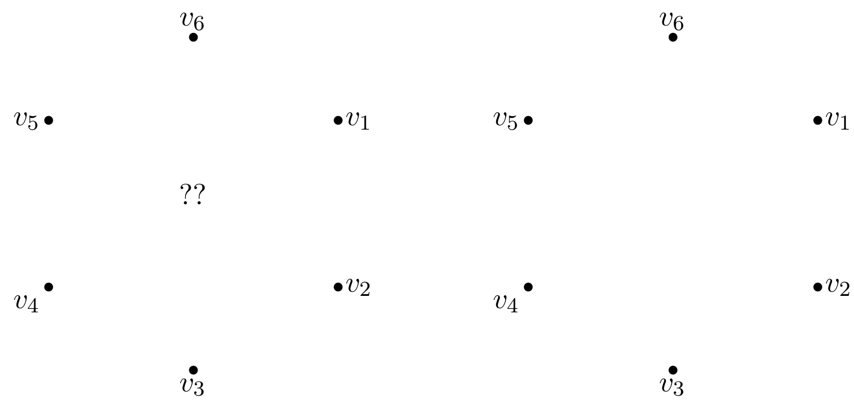
(6) a 1-connection from v_1 to v_2 and

(7) a 1-connection from v_3 to v_4.

To see this, let $R = (v_4, v_1)$ and $S = (v_3, v_2)$, and consider the submatrix

$$\Lambda(R; S) = \det \begin{bmatrix} \Lambda(4,3) & \Lambda(4,2) \\ \Lambda(1,3) & \Lambda(1,2) \end{bmatrix}$$

The off-diagonal entries $\Lambda(1,3)$ and $\Lambda(4,2)$ are both non-zero (negative) by (4) and (5). By Theorem **??**, $\det \Lambda(R; S)$ cannot be negative, so both diagonal entries $\Lambda(1,2)$ and $\Lambda(4,3)$ must be non-zero, which implies the



(a) Paths unknown (b) Possible connections

Figure 3-7: Unknown Paths

connections of (6) and (7). In this second use of Theorem **??**, it is important that the pair $(R; S)$ be in circular order. Figure **??**b shows a possibility for connections through the graph that satisfy the conditions.

Example 3.5 Let $\Gamma = (G, \gamma)$ be a resistor network with six boundary nodes as in Figure **??**a. Suppose it is known that

(1) there is a 2-connection from (v_1, v_2) to (v_6, v_5), and

(2) there is a 2-connection from (v_2, v_3) to (v_5, v_4), but

(3) there is no 3-connection from $P = (v_1, v_2, v_3)$ to $Q = (v_6, v_5, v_4)$.

Then there must be:

(4) a 2-connection from (v_1, v_2) to (v_5, v_4) and also

(5) a 2-connection from (v_2, v_3) to (v_6, v_5).

The relevant submatrix of Λ is

$$\Lambda(P; Q) = \begin{bmatrix} \Lambda(1,6) & \Lambda(1,5) & \Lambda(1,4) \\ \Lambda(2,6) & \Lambda(2,5) & \Lambda(2,4) \\ \Lambda(3,6) & \Lambda(3,5) & \Lambda(3,4) \end{bmatrix}$$

With the obvious (and standard) notation for the 2×2 corner matrices, the six-term identity of Lemma **??** shows that:

$$\det \Lambda(P;Q) \cdot \Lambda(2,5) = \det[NW] \cdot \det[SE] - \det[NE] \cdot \det[SW]$$

The assumptions imply that $\det[NW] \neq 0$, $\det[SE] \neq 0$. Theorem **??** implies that $\det \Lambda(P;Q) = 0$, so $\det[NE]$ and $\det[SW]$ must both be nonzero, which implies the existence of the connections asserted in (3) and (5). Figure **??**b shows a possibility for paths which satisfy the given conditions.

3.8 Recovery of Conductances I

To say that removing an edge e from a graph G breaks the connection from $P \leftrightarrow Q$ means that P and Q are connected through G (possibly in many ways), but that P and Q are not connected through G', which is the graph G which results when edge e is removed. By Theorem **??**, this is equivalent to the two assertions that $\det \Lambda(P;Q) \neq 0$ and $\det \Lambda'(P;Q) = 0$.

An edge pq between a pair of adjacent boundary nodes is called a *boundary edge*. If p is a boundary node which is joined by an edge to only one other node r which is an interior node, the edge pr is called a *boundary spike*.

Corollary 3.15 BOUNDARY EDGE FORMULA. *Let $\Gamma = (G, \gamma)$ be a circular planar resistor network and let pq be a boundary edge of G. Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are two sequences of boundary nodes such that $(P';Q') = (p, p_1, \ldots, p_k; q, q_1, \ldots, q_k)$ is a circular pair that is $(k+1)$-connected, and deleting pq breaks the connection between P' and Q'. Then*

$$\gamma(p,q) = -\Lambda(p;q) + \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) \tag{3.14}$$

Proof: Let $\xi = \gamma(p,q)$. Consider $\det K(P' + I; Q' + I)$ as a linear function $F(z)$ of the first column z of $K(P' + I; Q' + I)$. This is the column corresponding to node $q \in Q'$. Thus $z = x + y$, where

$$x = \begin{bmatrix} -\xi \\ 0 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

Then $F(z) = F(x) + F(y)$. Since there is no connection from P' to Q' after pq is deleted, $F(y) = 0$, Therefore

$$\begin{aligned}\det K(P' + I; Q' + I) &= F(x) + F(y)\\ &= F(x)\\ &= -\xi \det K(P + I; Q + I)\end{aligned}$$

Taking the Schur complement of $K(I; I)$ in $K(P + I; Q + I)$, and using Lemma **??** gives

$$\det \Lambda(P'; Q') = -\xi \cdot \det \Lambda(P; Q)$$

The Schur complement of $\Lambda(P; Q)$ in $\Lambda(P'; Q')$ produces the 1×1 matrix consisting of the single number:

$$\Lambda(p; q) - \Lambda(p; Q) \cdot \Lambda(P; Q)^{-1} \cdot \Lambda(P; q)$$

The determinantal identity for Schur complements, equation **??**, shows that this value is $-\xi$. This gives the formula for $\gamma(p, q)$.

Example 3.6 Let $\Gamma = (G, \gamma)$ be a resistor network whose graph is the graph G of Figure **??**, and suppose the response matrix Λ is:

$$\Lambda = \begin{bmatrix} 2.4 & -0.4 & 0.0 & -0.8 & -1.2 \\ -0.4 & 1.9 & -1.0 & -0.2 & -0.3 \\ 0.0 & -1.0 & 2.0 & -1.0 & 0.0 \\ -0.8 & -0.2 & -1.0 & 3.6 & -1.6 \\ -1.2 & -0.3 & 0.0 & -1.6 & 3.1 \end{bmatrix} \tag{3.15}$$

Deleting edge v_4v_5 breaks the connection $P' \leftrightarrow Q'$, where $P = (v_4, v_2)$, and $Q = (v_5, v_1)$. Thus with $p = v_4$, $q = v_5$, $P = (v_2)$ and $Q = (v_1)$, equation **??** gives

$$\begin{aligned}\gamma(v_4, v_5) &= -\Lambda(4; 5) + \Lambda(4; 1)\Lambda(2; 1)^{-1}\Lambda(2, 5)\\ &= 1.6 + (-0.8)(-0.4)^{-1}(-0.3)\\ &= 1.6 - (0.6) = 1.0\end{aligned}$$

Corollary 3.16 Boundary spike formula. *Suppose Γ is a circular planar resistor network and pr is boundary spike joining a boundary node p to*

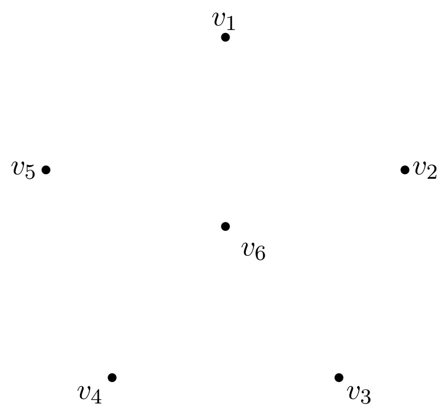


Figure 3-8: Graph G

an interior node r. Suppose that contracting pr to a single node breaks the connection between a circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$. Then

$$\gamma(pr) = \Lambda(p;p) - \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;p) \tag{3.16}$$

Proof: Let $\xi = \gamma(pr)$. Then $K(P+p+I;Q+p+I)$ has a submatrix $K(p,r;p,r)$ which has the form:

$$K(p,r;p,r) = \begin{bmatrix} \xi & -\xi \\ -\xi & \sigma \end{bmatrix}$$

The remaining entries of $K(P+p+I;Q+p+I)$ in both the row and the column corresponding to p are 0. Thus

$$\begin{aligned} \det K(P+p+I;Q+p+I) = \xi \det K(P+I;Q+I) \\ - \xi^2 \det K(P+I-r;Q+I-r) \end{aligned}$$

The last term is 0 since contracting pr breaks the connection from P to Q. If each of the other terms is divided by $\det K(I;I)$, and each quotient interpreted as the determinant of a Schur complement, the result is:

$$\det \Lambda(P+p;Q+p) = \xi \cdot \det \Lambda(P;Q)$$

Taking the Schur complement of $\Lambda(P;Q)$ in $\Lambda(P+p;Q+p)$ results in a 1 $\times$ 1 matrix with the single entry:

$$\Lambda(p;p) - \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;p)$$

The determinantal identity **??** shows that this value is $\xi = \gamma(pr)$.

Example 3.7 Again let G be the graph of Figure **??** with the response matrix Λ given by equation **??**. Contracting edge v_1v_6 in the graph G breaks the connection $P \leftrightarrow Q$, where $P = (v_5, v_4)$ and $Q = (v_2, v_3)$. Formula **??** gives

$$\begin{aligned}
\gamma(v_1v_6) &= \Lambda(1;1) - \Lambda(1;2,3)\Lambda(5,4;2,3)^{-1}\Lambda(5,4;1) \\
&= 2.4 - [-0.4, 0] \begin{bmatrix} -0.3 & 0.0 \\ -0.2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1.2 \\ -0.8 \end{bmatrix} \\
&= 2.4 - (-1.6) = 4.0
\end{aligned}$$

Chapter 4

Harmonic Functions

4.1 Harmonic Continuation

In this section, Kirchhoff's Laws will be used to construct γ-harmonic functions on a resistor network, in particular, γ-harmonic functions where the boundary values are prescribed at some of the boundary nodes, and the boundary currents are prescribed at some of the same nodes. This is not always possible, but when it is, it provides useful information about the network. First to be considered are rectangular networks, with nodes at (some of) the integer lattice points, as illustrated in Figure **??**.

Specifically, the rectangular graph $R(m, n)$ has nodes at points (i, j) for $0 \leq i \leq m+1$ and $0 \leq j \leq n+1$, with the four corner points $(0, 0)$, $(m+1, 0)$, $(m+1, n+1)$ and $(0, n+1)$ omitted. The edges are the horizontal or vertical line segments between adjacent nodes, excluding the edges on the lines $x = 0$, $x = m+1$, $y = 0$, and $y = n+1$. Thus $R(m, n)$ has $2n + 2m$ boundary nodes, mn interior nodes, and $n(m+1) + (n+1)m$ edges.

• The boundary nodes on the right-hand face are called East nodes, or simply E. These are the nodes (i, j) where $i = m+1$ and $1 \leq j \leq n$.

• The boundary nodes on the bottom face are called South nodes, or S. These are the nodes (i, j) where $j = 0$ and $1 \leq i \leq m$.

• The boundary nodes on the left-hand face are called West nodes, or W. These are the nodes (i, j) where $i = 0$ and $1 \leq j \leq n$.

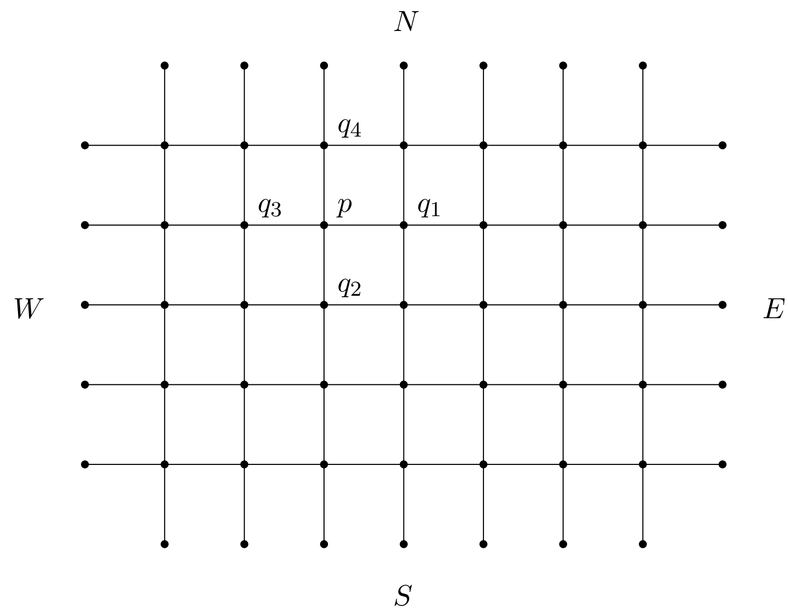


Figure 4-1: Rectangular graph

• The boundary nodes on the top face are called North nodes, or N. These are the nodes (i, j) where $j = n + 1$ and $1 \leq i \leq m$.

• The nodes of $R(m, n)$ along the vertical line $x = i$ are denoted by Y_i. Thus Y_0 is the west face W, and Y_{n+1} is the east face E.

The boundary "circle" for $R(m, n)$ passes through the nodes on the faces E, S, W, and N.

The graph $R(7, 5)$ in Figure **??** has 5 boundary nodes on each of the E and W faces, and 7 boundary nodes on each of the N and S faces.

Suppose u is a γ-harmonic function on a rectangular network, and p is an interior node with 4 neighboring nodes q_1, q_2, q_3 and q_4, as shown in Figure **??**. For such a configuration, equation **??** is a *5-point formula*:

$$\left(\sum_{j=1}^{4} \gamma(pq_j)\right) u(p) = \sum_{j=1}^{4} \gamma(pq_j)u(q_j) \tag{4.1}$$

If any four of these values of the function u are known, the fifth value is determined. In particular, if the values of $u(p)$, $u(q_2)$, $u(q_3)$ and $u(q_4)$

are all known, then the value of $u(q_1)$ will be known too. If the values of $u(q_2)$, $u(q_3)$ and $u(q_4)$ are known, and the current through q_3p is known, the value of $u(p)$ can be calculated by using Ohm's Law and then $u(q_1)$ can be calculated by the 5-point formula, **??**.

Observation 4.1 There is a similar statement when the points q_2, q_3, q_4 are replaced by the entire faces S, W and N, the node p is replaced by the set of interior nodes, and q_1 is replaced by the entire E face.

Suppose that u is a γ-harmonic function on a rectangular network whose values are given on the S, W, and N faces, and suppose also that the current is given on the W face. Then the value of u at all the nodes of Γ can be found by a process called *harmonic continuation* as follows.

(1) The values of u on $Y_0 = W$, the values of the horizontal conductors on the W face, the currents into Γ at the nodes of W and Ohm's Law, give the values of $u(1, j)$, for $1 \leq j \leq n$. Together with the given values, $u(1, 0)$ and $u(1, n+1)$, the values of u can be found for all nodes in Y_1.

(2) The values of $u(2, j)$, for $1 \leq j \leq n$ can be calculated from the values of u at the nodes in the vertical lines Y_0 and Y_1 and the 5-point formula **??**. Together with the given values $u(2, 0)$ and $u(2, n+1)$ on the boundary, the values of u can be found for all nodes in Y_2,

(3) Continuing in this way across Γ, the values of u at each of the nodes in each of the columns $Y_1, \ldots Y_n$ can be calculated. And last to be found are the values of u in Y_{n+1}, which are the E face nodes.

Example 4.1 Suppose that the conductance of each edge in the rectangular network of Figure **??**, has value 1 and that the boundary values for a γ-harmonic function u are 0 for all nodes in the S, W and N faces and the current on the W face is given by: $\phi(0, j) = (-1)^j$ for $1 \leq j \leq 5$. The γ-harmonic function u with this boundary data will have values $[\,0, +1, -1, +1, -1, +1,\, 0]$ on the nodes in column Y_1. The values of u are successively greater and greater in magnitude in columns $Y_2, \ldots, Y_7$, and the signs alternate vertically in each Y_i. The values of u at all the nodes of the network are given in Figure **??**. This γ-harmonic function has exponential growth in the x-direction, and sinusoidal behavior in the y-direction.

$$\begin{bmatrix}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 5 & 25 & 129 & 681 & 3652 & 19799 & 108151 \\
0 & -1 & -6 & -34 & -190 & -1057 & -5872 & -32607 & -181042 \\
0 & 1 & 6 & 35 & 202 & 1153 & 6574 & 36687 & 205438 \\
0 & -1 & -6 & -34 & -190 & -1057 & -5872 & -32607 & -181042 \\
0 & 1 & 5 & 25 & 129 & 681 & 3652 & 19799 & 108151 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{bmatrix}$$

Figure 4-2: Table of values of a harmonic function on $R(7,5)$

4.2 Recovering Conductances from Λ

Suppose that $\Gamma = (R, \gamma)$ is a resistor network whose underlying graph is a rectangular graph R. The conductances can be calculated from Λ_γ using the functions constructed by harmonic continuation. If boundary values for u are specified appropriately at some of the boundary nodes, and the boundary currents are specified at some of those nodes, harmonic continuation guarantees that the potential u is 0 throughout a zone of Γ. The response matrix is used to determine the boundary values for this function. The response matrix is used a second time to find the current into the boundary nodes, and finally Ohm's Law is used to calculate the values of the conductors.

The method for recovering the conductances will be illustrated first for a resistor network $\Gamma = (G, \gamma)$ whose underlying graph is the square network $G = R(5,5)$ which has 5 boundary nodes on each face shown in Figure **??**. The same considerations apply to any rectangular network. Starting at the top, the nodes on the E face are labeled, $p_1, p_2, \ldots, p_5$. Starting from the left, the nodes on the N face are labeled, $q_5, q_4, \ldots, q_1$. The node on the N face adjacent to p_1 is q_1.

There is a γ-harmonic function u which has value 0 for all nodes on the S, W and N faces except $u(q_1) = 1$, and the current is 0 on the W face. Harmonic continuation guarantees that there is such a function and the value of u is 0 at all interior nodes. The value of $u(p_1)$ is uniquely determined by the values of the conductors in Γ. This is what permits the calculation of the values of $\gamma(p_1 r)$ and $\gamma(q_1 r)$. (Incidentally, but not essential to the argument at this point, the values of u at p_2, p_3, p_4 and p_5 are all 0.) If the conductances of all edges in G were known, the value of $u(p_1)$ could be

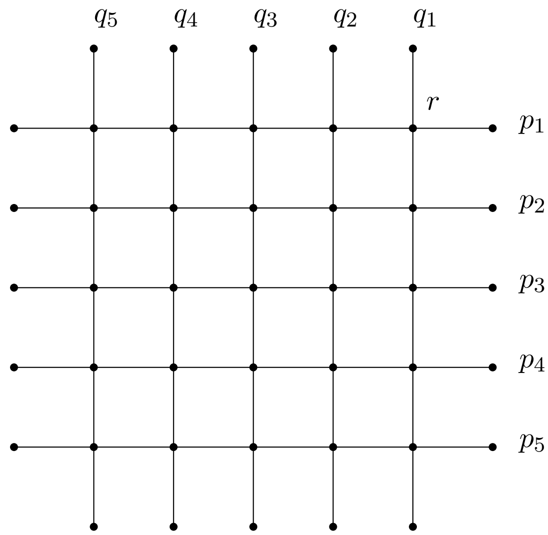


Figure 4-3: The graph $R(5,5)$

obtained by harmonic continuation starting from the nodes on the W face, and continuing across to the E face. Since the conductances are not yet known, the response matrix Λ_γ is used to find the value of $u(p_1)$ as follows. Λ_γ has the block form shown in Figure **??**. Here E stands for the 5 indices corresponding to the nodes on the E face, S stands for the 5 indices on the S face, etc. Thus the block $\Lambda(W;N)$ at position (W,N) is a 5×5 matrix which gives the current on the W face due to boundary values imposed on the N face. A similar block structure will be used for vectors of voltages and vectors of currents. For example the notation u_N stands for a vector of 5 values which are boundary voltages on the N face.

(1a) There is a potential for which the boundary values are the function y, where $y(q_1) = 1$, and for all other boundary nodes p, $y(p) = 0$. Using the notation for vectors and currents implied by the block structure of Λ, this means that $y_E = 0$, $y_S = 0$, $y_W = 0$, $y_N = [0,0,0,0,1]$ and $y = [y_E, y_S, y_W, y_N]$. The value of the current (at all nodes) is $\psi = \Lambda \cdot y$. The value of the current on the W face is $\psi_W = \Lambda(W;N) \cdot y_N$.

(1b) The submatrix $\Lambda(W;E)$ of Figure **??** gives the current on the W

$$\Lambda = \begin{array}{c|c|c|c|c|} & E & S & W & N \\ \hline E & \Lambda(E;E) & \Lambda(E;S) & \Lambda(E;W) & \Lambda(E;N) \\ \hline S & \Lambda(S;E) & \Lambda(S;S) & \Lambda(S;W) & \Lambda(S;N) \\ \hline W & \Lambda(W;E) & \Lambda(W;S) & \Lambda(W;W) & \Lambda(W;N) \\ \hline N & \Lambda(N;E) & \Lambda(N;S) & \Lambda(N;W) & \Lambda(N;N) \\ \hline \end{array}$$

Figure 4-4: Block structure of Λ for a rectangular graph

face due to boundary values imposed on the E face. The 5 nodes on the E face and the 5 nodes on the W face, are 5-connected, so by Theorem **??**, the matrix $\Lambda(W;E)$ is nonsingular, and there is a unique solution $x_E = [x_1, x_2, x_3, x_4, x_5]$ to the 5×5 linear system

$$\Lambda(W;E) \cdot x_E + \psi_W = 0$$

(1c) Let u be the potential on Γ due to the boundary function x, where (in block form) $x = [x_E, 0, 0, x_N]$ and x_E is obtained above, $x_S = 0$, $x_W = 0$ and $x_N = [0, 0, 0, 0, 1]$ on the N face. The network response is the current $\phi = \Lambda \cdot x$. On the W nodes the current is

$$\phi_W = \Lambda(W;E) \cdot x_E + 0 + 0 + \Lambda(W;N) \cdot x_N = 0$$

Using the fact that the current is 0 on the W face, harmonic continuation starting from the W face shows that the value of u is 0 for all the interior nodes. The current due to the boundary function x is $\phi = \Lambda \cdot x$. The value of the current ϕ at node q_1 gives the current across $q_1 r$. The values u are $u(q_1) = 1$, and $u(r) = 0$, so by Ohm's Law

$$\begin{aligned} \phi(q_1) &= \gamma(q_1 r)(u(q_1) - u(r)) \\ &= \gamma(q_1 r)(1 - 0) \end{aligned}$$

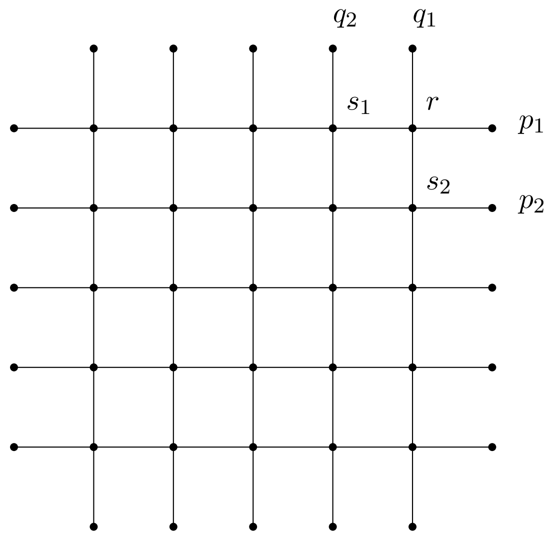


Figure 4-5: $R(5,5)$

The value of $\gamma(q_1 r)$ is calculated as follows.

$$\begin{aligned}\gamma(q_1 r) &= \phi(q_1) \\ &= \Lambda(q_1;E)x_E + \Lambda(N;q_1)x_N \\ &= -\Lambda(q_1;E)\cdot\Lambda(W;E)^{-1}\cdot\Lambda(W;q_1) + \Lambda(q_1;q_1)\end{aligned}$$

This is the same as the boundary spike formula **??**. The value of the conductors at each of the corners is calculated in the same way.

(2) The next step is to find the values of the conductors $\gamma(q_2 s_1)$, $\gamma(s_1 r)$, $\gamma(s_2 p_2)$ and $\gamma(r s_2)$ in Figure **??**.

(2a) There is a potential with boundary function y, where $y(q_2) = 1$ and $y = 0$ for all other boundary nodes. Thus $y_E = 0$, $y_S = 0$, $y_W = 0$, $y_N = [0,0,0,1,0]$ and $y = [y_E, y_S, y_W, y_N]$. The network response at all boundary nodes due to y is the current $\psi = \Lambda \cdot y$. The current on the W face is $\psi_W = \Lambda(W;N)\cdot y_N$. Since $\Lambda(W;E)$ is non-singular, the 5×5 linear system

$$\Lambda(W;E)\cdot x_E + \psi_W = 0$$

has a unique solution vector x_E. There is a potential u for which the boundary function is x where $x_N = y_N$ and $x = [x_E, 0, 0, y_N]$. The current due to the potential u is $\phi = \Lambda \cdot x$. The current on the W nodes

$$\phi_W = \Lambda(W;E) \cdot v_E + 0 + 0 + \Lambda(W;N) \cdot x_N = 0$$

Harmonic continuation starting from the W face shows that the value of u is 0 at all interior nodes, except at node r. The value of u is specified to be 1 at q_2. The value of the current at node q_2 gives the current across conductor $q_2 s_1$. Ohm's Law is:

$$\phi(q_2) = \gamma(q_2 s_1)(u(q_2) - 0)$$

This gives the value of $\gamma(q_2 s_1)$.

$$\begin{aligned}\gamma(q_2 s_1) &= \phi(q_2) \\ &= \Lambda(q_2;E)x_E + \Lambda(N;q_2)x_N \\ &= -\Lambda(q_2;E)\Lambda(W;E)^{-1}\Lambda(W;q_2) + \Lambda(q_2;q_2)\end{aligned}$$

This is another case of the boundary spike formula of Corollary **??**.

(2b) Furthermore, $\phi(q_1)$ is the value of the current across conductor $q_1 r$. The potential at q_1 is 0, the conductance $\gamma(q_1 r)$ has been calculated, so the value of $u(r)$ can be found by Ohm's Law.

$$\phi(q_1) = \gamma(q_1 r)(0 - u(r))$$

The current $c(s_1 r)$ across $s_1 r$ must be the same as the current across $q_2 s_1$, which is $\phi(q_2)$. The value of $\gamma(s_1 r)$ can be calculated by Ohm's Law.

$$\begin{aligned}\phi(q_2) &= \gamma(s_1 r)(u(s_1) - u(r)) \\ &= \gamma(s_1 r)(0 - u(r))\end{aligned} \tag{4.2}$$

(3) The response matrix can be used in a similar way to find the conductances of the edges at level three, that is, the values of the conductors $\gamma(q_3, t_1)$ $\gamma(t_1, s_1)$ $\gamma(s_1, t_2)$ $\gamma(t_2, s_2)$ $\gamma(s_2, t_3)$ $\gamma(t_3, p_3)$ in Figure **??**. The first step is to start with a potential with boundary values x, where $x(q_3) = 1$, and $x(p) = 0$ for all other boundary nodes. Thus $x_E = 0$, $x_S = 0$, $x_W = 0$, $x_N = [0, 0, 1, 0, 0]$ and $x = [x_E, x_S, x_W, x_N]$. The rest of the calculation is similar to the calculation at level (2) above. For $R(5, 5)$, the recovery of the conductances is complete after five levels.

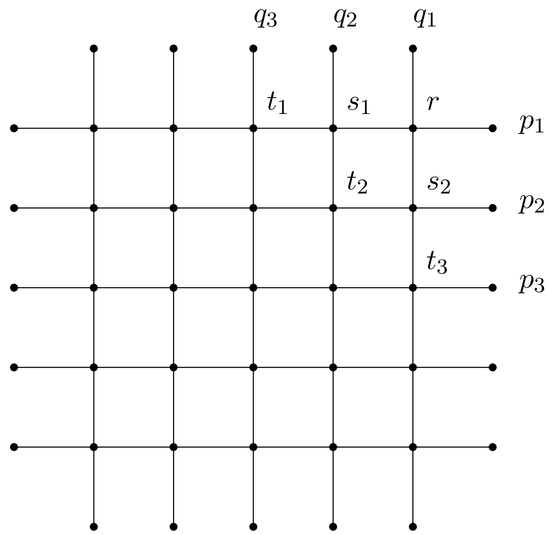


Figure 4-6: $R(5,5)$

4.3 Special Functions on Networks

Suppose $\Gamma = (G, \gamma)$ is circular planar resistor network with response matrix Λ, and $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is a circular pair of sequences of boundary nodes. If there is a k-connection from P to Q, Theorem **??** implies that the sub-matrix $\Lambda(P;Q)$ of Λ is non-singular. This fact will be used to construct special γ-harmonic functions on Γ. These functions are obtained by imposing conditions on u, some of which are boundary values, and some of which are boundary currents, similar to the conditions which were used for rectangular networks. These functions will be used extensively in the recovery algorithm for the well-connected critical graphs of Section **??**.

Lemma 4.1 *Let Γ be a circular planar resistor network with n boundary nodes. Let $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ be two sequences of boundary nodes which are in disjoint arcs, and suppose there is a k-connection from P to Q. Let R be the set of boundary nodes not in Q. Suppose the boundary values are specified to be 0 at the nodes of R and suppose k real numbers $c_1, \ldots, c_k$ are chosen to be boundary currents at the nodes of P. Then there* *is a unique γ-harmonic function u on Γ with the specified boundary data.*

Proof: Since there is a k-connection from P to Q, Theorem **??** implies that the sub-matrix $\Lambda(P;Q)$ of Λ is non-singular. Let $c = [c_1, c_2, \ldots, c_k]$, and take $x = [x_1, x_2, \ldots, x_k]$ to be the solution to the matrix equation

$$\Lambda(P;Q)x = c$$

Let u be the potential on G with boundary values $[x_1, x_2, \ldots, x_k]$ at nodes $q_1, q_2, \ldots, q_k$, and 0 at all other boundary nodes. This function u on G has the specified currents and boundary values.

Example 4.2 Refer to Figure **??**. Suppose $\Gamma = (G, \gamma)$ is a resistor network whose underlying graph is a circular planar graph G with n boundary nodes. Suppose $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ are a circular pair which are 3-connected. According to Lemma **??**, there is a potential u on Γ with boundary current ϕ, where $\phi(p_1) = +1$, $\phi(p_2) = -1$, $\phi(p_3) = +1$, and where the value of u is 0 at all boundary nodes except q_1, q_2 and q_3.

(1) The current of $(+1)$ at p_1 and the value $u(p_1) = 0$ together imply that there is a node r_1 adjacent to p_1 of negative potential, and another node adjacent to that of yet more negative potential, leading eventually to a boundary node of negative potential. Call this path α_1.

(2) The current of (-1) at p_2 and the value $u(p_2) = 0$ together imply that there is a node r_2 adjacent to p_2 of positive potential, another node adjacent to that of yet higher positive potential, leading to a boundary node of positive potential. Call this path α_2.

(3) The current of $(+1)$ at p_3 and the value $u(p_3) = 0$ together imply that there is a node r_3 adjacent to p_3 of negative potential, and another node adjacent to that of yet more negative potential, leading to a boundary node of negative potential. Call this path α_3.

The path α_1 cannot intersect the path α_2 because the potential is negative at all interior nodes of α_1 and positive at all interior nodes of α_2. Similarly, the path α_2 cannot intersect the path α_3 because the potential is positive on α_2 and negative on α_3. The only boundary nodes of non-zero potential are q_1, q_2 and q_3. Therefore

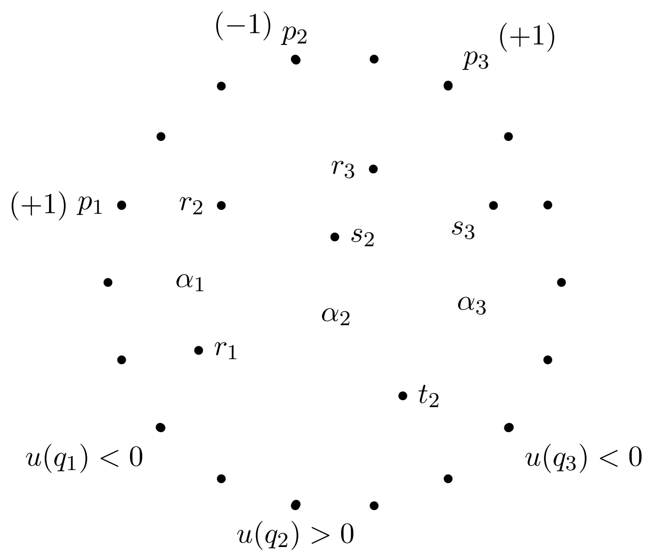


Figure 4-7: Flow paths across a network

(1) The path α_1 must connect p_1 to q_1. On all nodes of α_1 except p_1, the potential will be negative; the potential at q_1 is negative.

(2) The path α_2 must connect p_2 to q_2. On all nodes of α_2 except p_2, the potential will be positive; the potential at q_2 is positive.

(3) The path α_3 must connect p_3 to q_3. On all nodes of α_3 except q_3, the potential will be negative; the potential at q_3 is negative.

A drawing of these paths is shown in Figure **??**. The arrows indicate direction of current flow, which is the direction of decreasing potential. The currents at nodes p_1, p_2 and p_3 are indicated by (+1), (−1) and (+1) respectively.

There may be more than one path $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ joining P to Q. The argument shows only the existence of at least one such path α.

Observation 4.2 Suppose $\Gamma = (G, \gamma)$ is a resistor network whose underlying graph is a circular planar graph G with n boundary nodes. Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are a circular pair which are k-connected. Suppose y is a boundary function which is 0 except at the nodes in Q. Let ϕ be the boundary current due to y and suppose the values of ϕ alternate in sign at the nodes in P. Then, just as in Example **??**, the values of y must alternate in sign at the nodes in Q. Specifically if $(-1)^{i+1}\phi(p_i) > 0$, then it must happen that $(-1)^i y(q_i) > 0$. In matrix form, this says that if c alternates in sign, then the solution x to

$$\Lambda(P; Q) \cdot x = c$$

must alternate in sign also. In the terminology of [**?**], this says that if $\Lambda(P; Q)$ is non-singular, then $\Lambda(P; Q)^{-1}$ has the alternating property. For the circular planar graphs in [**?**], the alternating property was established geometrically, and then used to prove that the existence of a connection $P \leftrightarrow Q$ implies the non-singularity of the matrix $\Lambda(P; Q)$. In this text we have given a purely algebraic proof of Theorem **??**.

More general boundary conditions may also be imposed, as follows.

Theorem 4.2 *Let Γ be a circular planar resistor network with n boundary nodes. Let $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ pair of sequences of boundary nodes which are in disjoint arcs of the boundary circle, and suppose that there is a k-connection from P to Q. Let R be the set of boundary nodes not in Q. Suppose $n-k$ boundary values $\{b_i\}$ are specified arbitrarily at the nodes v_i of R and suppose k real numbers $\{c_i\}$ are specified to be boundary currents at the nodes v_i of P. Then there is a unique γ-harmonic function u on Γ with this boundary data.*

Proof: Suppose Λ is the response matrix for Γ. Take $y = [y_1, \ldots, y_n]$ to be the boundary function, where $y_i = b_i$ for $v_i \in R$, and $y_i = 0$ for $v_i \notin R$. Let $\psi = \Lambda \cdot y$, and then take $d = [d_1, \ldots, d_k]$ for $1 \leq i \leq k$ to be the set of boundary currents, where $d_i = c_i - \psi(q_i)$ for $q_i \in Q$. Lemma **??** implies that there is a boundary function z with $z_i = 0$ for $v_i \in R$, and the potential due to z has current d_i at the nodes $q_i \in Q$. Let x be the boundary function where $x = y + z$. The potential u due to x has the specified boundary data.

4.4 Special Functions on G_{4m+3}

Let $n = 4m + 3$, and let $\Gamma = (G_n, \gamma)$ be a circular planar resistor network whose underlying graph is the well-connected graph G_n with $4m + 3$ boundary nodes. Theorem **??** shows that there are special γ-harmonic functions on Γ. Recall the description of the circular planar graph G_n. There are n rays $\rho_0, \ldots, \rho_{n-1}$ originating at the origin and making angles $\theta_0, \ldots, \theta_{n-1}$ measured clockwise from the first ray ρ_0, with $0 = \theta_0 < \ldots < \theta_{n-1} < 2\pi$. (By convention, $\rho_n = \rho_0$.) The circles have radii $1, \ldots, m$. The node in G_n which is the intersection of circle of radius i with ray ρ_j is denoted (i, j). The boundary nodes are $v_j = (m + 1, j)$ for $1 \le j \le n$. By convention, the boundary node $v_0 = (m + 1, 0) = (m + 1, n) = v_n$.

There is a γ-harmonic function f on G_{4m+3} with boundary current ϕ which has the following boundary data:

$$\begin{aligned} f(v_0) &= 1 \\ f(v_j) &= 0 \quad \text{for} \quad 2m+2 \le j \le 4m+2 \\ \phi(v_j) &= 0 \quad \text{for} \quad 2m+2 \le j \le 4m+2 \end{aligned} \tag{4.3}$$

This function f has value 0 throughout a zone Z of G_{4m+3}. Specifically $f(i, j) = 0$, except for those nodes (i, j) for which both of the following conditions hold.

(1) $1 \le i \le m + 1$

(2) $m + 1 - i \le j \le m + i$

Harmonic continuation starting from the boundary nodes $v_{2m+2}, \ldots v_{4m+2}$ shows that f is 0 throughout a wedge, with vertex at $(0, 0)$ and whose outer vertices are $v_{2m+2}, \ldots, v_{4m+2}$. Another use of harmonic continuation, both clockwise and counterclockwise from the sides of the wedge, shows that f is 0 throughout the zone Z.

The sign of $f(p)$ can be determined for all nodes p in G. The zone of 0's for f includes the nodes $(m, 0)$ and $(m - 1, 1)$. The given value of the function is $f(m+1, 0) = 1$. The averaging property of a γ-harmonic function (Chapter **??**, equation **??**) implies that $f(m, 1)$ must be negative. Similarly, $f(m - 1, 2)$ must be positive, $f(m - 2, 3)$ must be negative, etc. This is summarized in Proposition **??**.

Proposition 4.3 *The harmonic function f on G_{4m+3} with boundary data given by* **??** *satisfies the following.*

(1) For $0 \le j \le m$, and j even,

$$0 = f(m-j,j) < \ldots < f(i,j) < \ldots < f(m+1,j)$$

(2) For $0 \le j \le m$, and j odd,

$$0 = f(m-j,j) > \ldots > f(i,j) > \ldots > f(m+1,j)$$

(3) For $m+1 \le j \le 2m$, and j even,

$$0 = f(2m+1-j,j) < \ldots < f(i,j) < \ldots < f(m+1,j)$$

(4) For $m+1 \le j \le 2m$, and j odd,

$$0 = f(2m+1-j,j) > \ldots > f(i,j) > \ldots > f(m+1,j)$$

(5) For any edge pq with at least one of the endpoints not in the zone of 0's, $f(p) - f(q) \neq 0$.

Example 4.3 This is illustrated in Figure **??** for the graph G_{11}. $f(v_0) = +1$. The nodes where the function f must be positive are indicated by $+$, and the nodes where f must be negative are indicated by $-$. At all other nodes, the value of the function f is 0. Thus the sign of the potential f is known at all nodes of G. By (1) to (4), the direction of current flow across all edges of G is also known.

The function f constructed above is denoted $f^{(0)}$. There are similar functions $f^{(1)}, \ldots, f^{(n-1)}$, obtained by rotating the graph through angles θ_1, $\ldots$, θ_{n-1}. Specifically, the boundary values and boundary currents $\phi^{(k)}$ for the function $f^{(k)}$ will be

$$\begin{aligned} f^{(k)}(v_k) &= 1 \\ f^{(k)}(v_{k+j}) &= 0 \quad \text{for} \quad 2m+2 \le j \le 4m+2 \\ \phi^{(k)}(v_{k+j}) &= 0 \quad \text{for} \quad 2m+2 \le j \le 4m+2 \end{aligned}$$

There is another family of functions $g^{(k)}$ similar to the functions $f^{(k)}$, obtained by reversing the directions around the boundary circle. For each

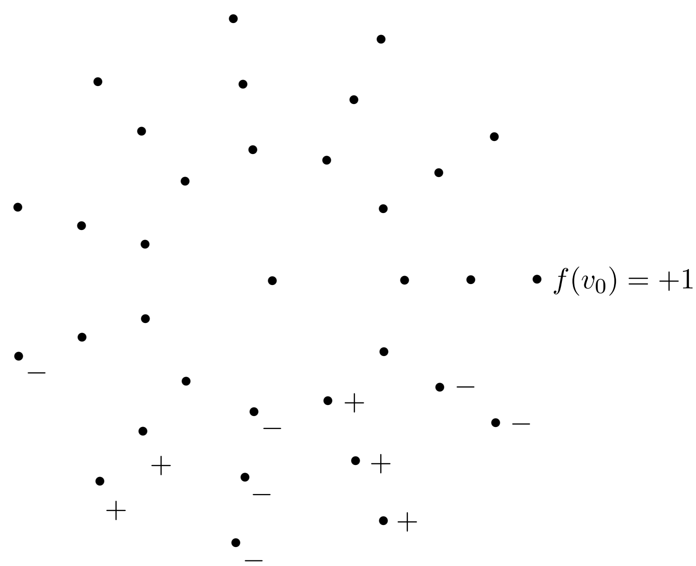


Figure 4-8: Zone of zeros in G_{11}

$0 \leq k \leq 4m+2$, the boundary values and boundary currents $\psi^{(k)}$ for $g^{(k)}$ are

$$\begin{array}{rcl} g^{(k)}(v_k) & = & 1 \\ g^{(k)}(v_{k+j}) & = & 0 \quad \text{for} \quad 1 \leq j \leq 2m+1 \\ \psi^{(k)}(z_{k+j}) & = & 0 \quad \text{for} \quad 1 \leq k \leq 2m+1 \end{array}$$

These special γ-harmonic functions $f^{(j)}$ and $g^{(j)}$ will be used in the recovery algorithm of Section **??**. These same special γ-harmonic functions $f^{(j)}$ and $g^{(j)}$ will be used in Section **??** to show that the differential of the map

$$L : (R^+)^N \longrightarrow R^{n^2}$$

which takes γ to Λ_γ is non-singular.

4.5 Recovery of Conductances II

Let m be a fixed non-negative integer. Let $n = 2m+1$, and let $\Gamma = (G, \gamma)$ be a circular planar resistor network whose underlying graph is the circular planar graph G_n described in Chapter 2. Suppose the response matrix $\Lambda = \Lambda_\gamma$ is given. The response matrix Λ is used to find the boundary values for the special γ-harmonic functions described in Section **??**. These boundary values x and the corresponding currents $\phi = \Lambda x$ will be used in an algorithm for computing the conductance $\gamma(pq)$ for each edge in G_n.

In our notation $\Lambda(P; n)$ is the vector of currents at the nodes of P due to a potential of 1 at node v_n. Since G_n is well-connected, and P and Q are in disjoint arcs of the boundary circle, Theorem **??**, and Corollary **??** imply that $\Lambda(P; Q)$ is non-singular. Hence there is a unique solution for the vector $x_Q = [x_1, \ldots, x_m]$ in the matrix equation

$$\Lambda(P; Q)x_Q + \Lambda(P; n) = 0$$

Let $x = [x_1, x_2, x_3, \ldots, x_m, 0, \ldots, 1]$. This is an explicit calculation of the boundary values x and boundary current $\phi = \Lambda x$ for the function f of Section **??**. The potential due to x is the function f with boundary current ϕ for which

$$\begin{aligned} f(v_n) &= 1 \\ f(v_j) &= 0 \quad \text{for} \quad m+1 \le j \le 2m \\ \phi(v_j) &= 0 \quad \text{for} \quad m+1 \le j \le 2m \end{aligned} \tag{4.4}$$

The algorithm for computing conductances in G_n proceeds inwards by levels. For each boundary node v_j, let t_j be the interior node adjacent to v_j. Harmonic continuation shows that $f(t_n) = 0$. Therefore

$$\begin{aligned} \phi(v_n) &= \gamma(v_n t_n)(f(v_n) - f(t_n)) \\ &= \gamma(v_n t_n)(1 - 0) \end{aligned}$$

Thus

$$\begin{aligned} \gamma(v_n t_n) &= \phi(v_n) \\ &= \Lambda \cdot x(v_n) \\ &= \Lambda(n; Q)x_Q + 0 + \Lambda(n; n)x_n \\ &= -\Lambda(n; Q)\Lambda(P; Q)^{-1}\Lambda(P, n) + \Lambda(n; n) \end{aligned}$$

This is the boundary spike formula **??**. There is a similar formula for each of the boundary spikes. Let $P^{(j)} = (v_{m+1+j}, \ldots, v_{2m+j})$ and $Q^{(j)} = (v_{1+j}, \ldots, v_{m+j})$. Then

$$\gamma(v_j t_j) = -\Lambda(j; Q^{(j)}) \cdot \Lambda(P^{(j)}; Q^{(j)})^{-1} \cdot \Lambda(P^{(j)}; j) + \Lambda(j; j)$$

Observation 4.3 This shows that using harmonic continuation to produce a zone of 0's in Γ results in a calculation of the conductances of boundary spikes which is the same as that of formula **??**, which was obtained by taking Schur complements. If the network had boundary edges instead of boundary spikes, harmonic continuation could also be used to produce a zone of 0's resulting in a calculation of the conductance of each boundary edge which is the same as the boundary edge formula **??**.

After calculating all the conductances $\gamma(v_j t_j)$, the boundary values x and the boundary current $\phi = \Lambda x$ for the function f are used again to calculate conductances at the next level. For each edge rs in G_n, let $c(rs)$ denote the current from r to s. In our notation, $v_0 = v_n$ and $t_0 = t_n$. The current $c(t_0 t_1)$ across $t_0 t_1$ is the same as the current $c(v_0 t_0)$ across $v_0 t_0$ because the other neighbors of t_0 all have potential 0. The value of $f(t_1)$ can be calculated from $x_1 = f(v_1)$, the previously computed value of $\gamma(v_1 t_1)$ and the current $\phi(v_1)$. Thus

$$\phi(v_1) = \gamma(v_1 t_1)(f(v_1) - f(t_1))$$

$$\begin{aligned} \phi(v_0) &= c(v_0 t_0) \\ &= c(t_0 t_1) \\ &= \gamma(t_0 t_1)(f(t_0) - f(t_1)) \\ &= \gamma(t_0 t_1)(0 - f(t_1)) \end{aligned}$$

The result is

$$\gamma(t_0 t_1) = (-\frac{1}{f(t_1)})\phi(v_0)$$

The conductances $\gamma(t_j, t_{j+1})$ can be computed for $j = 1, \ldots, 2m+1$, in exactly the same way, using the functions $f^{(j)}$, and the known values of $\gamma(v_k t_k)$

For each $j = 1, \ldots, n$, let s_j be the interior node on ray j adjacent to t_j. The radial conductors $\gamma(t_j s_j)$ are calculated next. The value of f at all nodes t_j on the circle of radius m can be calculated from the boundary values x, the boundary current ϕ and the values of $\gamma(v_j t_j)$. The zone of 0's for f includes s_1, so the calculation of $\gamma(t_1 s_1)$ is similar to that of $\gamma(v_n t_n)$. Ohm's Law gives the current across edges $(v_1 t_1)$, $(t_n t_1)$, and $(t_1 t_2)$. Using Kirchhoff's Law, $f(s_1) = 0$ and $\gamma(t_1 s_1)$ is calculated by Ohm's Law. Similarly, conductances for all radial edges $t_j s_j$ are found using the boundary functions and the boundary currents for the special functions $f^{(j)}$ of Section **??**. The calculation of the circular conductances at radius $m - 1$, similar to the calculation of the circular conductances at radius m. Continuing inwards, the conductances of all edges in G_n are computed.

There is a similar algorithm for calculating the conductances in a resistor network $\Gamma = (G_n, \gamma)$ when n is an even integer. The conductances of the boundary edges are first calculated by the boundary edge formula **??**. Then the calculation proceeds just as for the case n odd.

This algorithm shows that the response matrix Λ_γ uniquely determines the conductivity γ. The algorithm also shows that the value of each conductance can be calculated by a rational algebraic expression that never involves division by 0. This is summarized in the following Theorem.

Theorem 4.4 *let Γ be a circular network whose underlying graph is the well-connected critical graph G_n. The map which sends the conductivity function γ to the response matrix Λ_γ is 1 - 1. Let γ and μ be two conductivities on G_n. If Λ_γ is sufficiently near to Λ_μ, then γ will be near to μ.*

The method of special functions can also be used to calculate the conductances of any circularly symmetric planar resistor network, provided that there are sufficiently many radial lines, in particular if there are m concentric circles, and the number of radial lines is n where $n \geq 4m+3$. The nodes of G are the intersection points of $m+1$ concentric circles and n rays, originating from the origin. The edges of G are the radial lines joining adjacent nodes on the rays, and the circular arcs joining adjacent nodes on all circles except the outer one. The recovery of the conductances for such a network is very similar to the recovery for the networks whose underlying graph is of the form G_{4m+3}. The uniqueness and the continuity of the inverse also hold for such networks.

4.6 The Differential of L

Let n be an integer with $n \geq 3$ and $G = G_n$ be the well-connected circular graph constructed in Chapter **??**. The number of edges in G_n is $N = n(n-1)/2$. For each conductivity function γ on G_n, let Λ_γ be the response matrix. Let $B_\gamma(\cdot\ ,\ \cdot)$ be the bilinear form in n variables as defined in Chapter **??**. For each pair of functions x and y defined on the boundary nodes of G_n,

$$B_\gamma(x, y) =< y,\, \Lambda_\gamma x >$$

Let $\mathcal{B}(n)$ be the space of bilinear forms in n variables. Then let

$$L\ : (R^+)^N \to \mathcal{B}(n)$$

be the function given by $L(\gamma) = B_\gamma(\ .\ ,\ .)$.

The following notation is used in the computation of the differential of L. If σ is a function defined on the edges of G, then

- $\sigma_{i,j}$ will stand for $\sigma(e)$ if there is an edge e joining v_i to v_j, and $\sigma_{i,j} = 0$ if there is no edge joining v_i to v_j. In particular, $\gamma_{i,j} = \gamma(e)$ if there is an edge e joining v_i to v_j. and $\gamma_{i,j} = 0$ if there is no edge joining v_i to v_j.

If f is a function defined on the nodes of G, then

- $f_i = f(v_i)$ is the value of f at the node v_i.
- $\nabla_{i,j} f = f_i - f_j$.
- $\phi_f(p)$ is the current into the network at node p.

For each pair of functions x and y defined on the boundary nodes of G, let u and w be the potentials due to x and y respectively. Then,

$$\begin{aligned} B_\gamma(x, y) &=< y, \Lambda x > \\ &= \sum \gamma_{i,j}(u_i - u_j)(w_i - w_j) \end{aligned}$$

Let κ be a real-valued function defined on the N edges of G, and let t be a real parameter sufficiently small so that $\gamma + t\kappa$ is positive on all the edges of G. Denote by u_t and w_t the $(\gamma + t\kappa)$-harmonic functions with boundary values x and y respectively. Then $u_t = u + \delta u_t$ and $w_t = w + \delta w_t$, where δu_t and δw_t are functions defined on the nodes of G which are 0 on the boundary nodes. Then

$$\begin{aligned}
B_{\gamma+t\kappa}(x, y) &= \sum (\gamma_{i,j} + t\kappa_{i,j})(\nabla_{i,j} u + \nabla_{i,j} \delta u_t)(\nabla_{i,j} \delta w + \nabla_{i,j} \delta w_t) \\
&= \sum \gamma_{i,j} (\nabla_{i,j} u)(\nabla_{i,j} w) + t \sum \kappa_{i,j} (\nabla_{i,j} u)(\nabla_{i,j} w) \\
&+ \sum \gamma_{i,j} \left((\nabla_{i,j} u)(\nabla_{i,j} \delta w_t) + (\nabla_{i,j} \delta u_t)(\nabla_{i,j} w) + (\nabla_{i,j} \delta u_t)(\nabla_{i,j} \delta w_t) \right) \\
&+ t \sum \kappa i, j \left((\nabla_{i,j} u)(\nabla_{i,j} \delta w_t) + (\nabla_{i,j} \delta u_t)(\nabla_{i,j} w) + (\nabla_{i,j} \delta u_t)(\nabla_{i,j} \delta w_t) \right)
\end{aligned}$$

Now

$$\begin{aligned}
\sum \gamma_{i,j} \nabla_{i,j} u \nabla_{i,j} \delta w_t &= \sum \delta w_t(p) \phi_u(p) = 0 \\
\sum \gamma_{i,j} \nabla_{i,j} \delta u_t \nabla_{i,j} w &= \sum \delta u_t(p) \phi_w(p) = 0
\end{aligned}$$

since $\phi_u(p) = 0 = \phi_w(p)$ for all $p \in \text{int } G$, and $\delta w_t(p) = 0 = \delta u_t(p)$ for all $p \in \partial G$.

If x is any function defined on the boundary nodes, and u is the potential due to x, the Kirchhoff matrix can be used to calculate the values of u at each interior node. Suppose the Kirchhoff matrix K for (G, γ) is:

$$K = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

where $D = K(I; I)$ is the block corresponding to the interior nodes of G. By formula **??**, the values of u at the interior nodes of G are:

$$u(p) = [-D^{-1} B^T x](p)$$

Similarly, let K_t be the Kirchhoff matrix for the conductivity $\gamma + t\kappa$ on G and suppose the block structure for K_t is:

$$K_t = \begin{bmatrix} A_t & B_t \\ B_t^T & D_t \end{bmatrix}$$

The values of u_t at the interior nodes of G are:

$$u_t(p) = [-D_t^{-1} B_t^T x](p)$$

The matrix $[-D_t^{-1}B_t^T]$ is a rational function of t, hence differentiable, with

$$D_0^{-1}B_0^T = -D^{-1}B^T$$

Therefore

$$D_t^{-1}B_t^T = D^{-1}B^T + tE_t$$

where $\lim_{t\to 0} E_t = 0$. Thus

$$D_t^{-1}B_t^T x = D^{-1}B^T x + tE_t x$$

This implies that $\delta u_t = t\tilde{u}_t$ where $\lim_{t\to 0} \tilde{u}_t = \tilde{u}_0$. Thus

$$\begin{aligned} B_{\gamma+t\kappa}(x,y) = \sum \gamma_{i,j}\nabla_{i,j}u\nabla_{i,j}w + t\sum \kappa_{i,j}\nabla_{i,j}u\nabla_{i,j}w \\ + t^2\sum \gamma_{i,j}\nabla_{i,j}\tilde{u}_t\nabla_{i,j}\tilde{w}_t \\ + t^2\sum \kappa_{i,j}\left(\nabla_{i,j}u\nabla_{i,j}\tilde{w}_t + \nabla_{i,j}\tilde{u}_t\nabla_{i,j}w + t\nabla_{i,j}\tilde{u}_t\nabla_{i,j}\tilde{w}_t\right) \end{aligned}$$

Hence

$$\frac{d}{dt}B_{\gamma+t\kappa}(x,y)\Big|_{t=0} = \sum \kappa_{i,j}\nabla_{i,j}u\nabla_{i,j}w \tag{4.5}$$

We return to the standard notation $f(p)$ for the value of a function f at node p, and $\kappa(pq)$ for the value of κ on the edge pq.

Lemma 4.5 *Let G_n be the circular network of Chapter 2. Let κ be any real-valued function on the edges of G_n. Suppose that for all γ-harmonic functions u and w, that*

$$\sum_{pq\in E} \kappa(pq)(u(p)-u(q))(w(p)-w(q)) = 0$$

Then κ is identically 0.

Proof: The proof will be given for n of the form $4m+3$; the proof for the other values of n is similar, but the indexing of the nodes is different. The nodes of G_n on the boundary circle (of radius $m+1$) are labeled $v_j = (m+1, j)$, for $j = 0, \ldots, n-1$. The nodes on the circle of radius m are labeled $t_j = (m, j)$; the nodes on the circle of radius $m-1$ are labeled $s_j = (m-1, j)$. The edges of G_n are ordered by levels from the outside inwards. That is, the outermost spikes $v_i t_i$ are at level 0, the circular edges

$t_i t_{i+1}$ are at level 1, the radial edges $t_i s_i$ are at level 2, etc. There are $2m+1$ levels, with the radial edges $((0,0),(1,j))$ at the last level $2m$.

The special functions f and g constructed in Section **??** have the property that the product

$$\kappa(pq)(f(p)-f(q))(g(p)-g(q)) \neq 0$$

only for the edge $pq = v_0 t_0$. This implies that $\kappa(v_0 t_0) = 0$. Similarly, for each spike $e = v_j t_j$ in G_n, the special functions $f^{(j)}$ and $g^{(j)}$, have the property that the product

$$\kappa(pq)(f^{(j)}(p)-f^{(j)}(q))(g^{(j)}(p)-g^{(j)}(q)) \neq 0$$

only when pq is the edge $v_j t_j$. Thus $\kappa(v_j t_j) = 0$ for $j = 0, \ldots, n-1$. Next, consider an edge of the form $t_j t_{j+1}$. The special functions $f^{(j)}$ and $g^{(j+1)}$ have the property that the product

$$\kappa(pq)(f^{(j)}(p)-f^{(j)}(q))(g^{(j+1)}(p)-g^{(j+1)}(q)) \neq 0$$

only when pq is the edge $t_j t_{j+1}$ or is a boundary spike. Thus $\kappa(t_j t_{j+1}) = 0$ for $j = 0, \ldots, n-1$.

More generally, for each edge e in G_n there is a pair of special functions $f^{(j)}$ and $g^{(k)}$, with the property that the product

$$\kappa(pq)(f^{(j)}(p)-f^{(j)}(q))(g^{(k)}(p)-g^{(k)}(q)) \neq 0$$

only when pq is the edge e or an edge pq which precedes e in the ordering. The proof that $\kappa(pq) = 0$ for all edges pq in G_n follows by induction using the ordering on the edges.

Let n be a positive integer, and G_n be the well-connected graph of Chapter **??**. For each conductivity function γ on G_n, let $L(\gamma) = B_\gamma(\,\cdot\,,\,\cdot\,)$ be the bilinear form in n variables defined by **??**. The space $\mathcal{B}(n)$, of bilinear forms in n variables, has dimension $N = n(n-1)/2$. L may be considered as a function from $(R^+)^N$ to $\mathcal{B}(n)$. It follows from the expression **??** and Lemma **??** that:

Theorem 4.6 *The differential of L is one-to-one.*

For each conductivity function γ on G_n, let Λ_γ be the response matrix. Let $\mathcal{P}$ be the set of pairs of integers (i, j) with $1 \leq i < j \leq n$, considered as positions of entries in Λ_γ. Suppose the N edges in G_n and the N positions in $\mathcal{P}$ are each ordered in some fixed way. For each $n \times n$ matrix A, let $P(A)$ be the point in R^N obtained by listing the entries of A in this fixed order. Let T be the function from $(R^+)^N$ to R^N, defined by $T(\gamma) = P(\Lambda_\gamma)$. It follows immediately from Theorem **??** and the relation between matrices and quadratic forms that:

Corollary 4.7 *The differential of T is one-to-one.*

Chapter 5

Characterization I

5.1 Properties of Response Matrices

We will give an explicit algebraic description of the set of all $n \times n$ matrices which are response matrices for conductivities on resistor networks $\Gamma = (G_n, \gamma)$ whose underlying graph is the well-connected graph G_n of Chapter **??**. The boundary nodes $v_1, v_2, \ldots, v_n$ are numbered clockwise around the boundary circle, with the convention that $v_n = v_0$; in general $v_{n+i} = v_i$.

Throughout this chapter, if A is a matrix, $A(i; j)$ refers to the entry in the (i, j)-position. More generally, if $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_m)$ are subsets of the rows and columns of A, then $A(P; Q)$ refers to the $k \times m$ sub-matrix of A obtained by taking the entries that are in rows $p_1, \ldots, p_k$ and columns $q_1, \ldots, q_m$ of A.

If A is an $n \times n$ matrix, the indices $\{1, 2, \ldots, n\}$ are in 1-1 correspondence with the points $\{v_1, v_2, \ldots, v_n\}$ on the boundary circle for the graph G_n. An ordered subset $S = (s_1, \ldots, s_k)$ of $\{1, \ldots, n\}$ is said to be in *circular order* if the corresponding points in $\{v_1, \ldots, v_n\}$ are in circular order on the unit circle as defined in Chapter **??**. If the numbers in the set $\{1, \ldots, n\}$ are identified with the corresponding points in the set $\{v_1, \ldots, v_n\}$, this means that $s_1, \ldots, s_k$ are in circular order if:

(1) $s_1 s_k$ is an arc of the circle.

(2) $0 = s_1 < s_2 < \ldots < s_k < 2\pi$ where the points s_i are measured by angles clockwise from s_1.

Following the definition in Chapter **??** of a circular pair of boundary nodes, a pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ of sequences of indices is said to be a *circular pair* if the sequence $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order. For example, let A be a 7×7 matrix, with index set $\{1, 2, \ldots, 7\}$, and let $P = (3, 4)$, $Q = (7, 5)$, $R = (2, 6)$. Then $(P;Q)$ is a circular pair, because $3, 4, 5, 7$ are in order in the set $3, 4, 5, 6, 7, 1, 2$. $(P;R)$ is a circular pair, because $3, 4, 6, 2$ are in order in the same set; but $(Q;R)$ is not a circular pair.

Recall from Chapter 3, that if $\Gamma = (G, \gamma)$ is a circular planar resistor network, the response matrix Λ_γ has the following properties.

$(P1)$ Λ_γ is symmetric; that is, $\Lambda_\gamma(i;j) = \Lambda_\gamma(j;i)$

$(P2)$ The sum of the entries in each row is 0.

$(P3)$ For each circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$,

$$(-1)^k \det \Lambda_\gamma(P;Q) \geq 0$$

Property (P3) above, when applied to the 1×1 subdeterminants of Λ_γ, says that each entry of Λ_γ is ≤ 0.

If G is well-connected as a graph, then for each circular pair $(P;Q)$, appearing under (P3), the value of $(-1)^k \det \Lambda_\gamma(P;Q)$ is *positive.* In particular, each entry of Λ_γ is strictly negative. We are led to consider all $n \times n$ matrices A which satisfy the following three conditions:

$(L1)$ The matrix A is symmetric; that is, $A(i;j) = A(j;i)$

$(L2)$ The sum of the entries in each row is 0

$(L3)$ For each circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$,

$$(-1)^k \det A(P;Q) > 0$$

For each non-negative integer n, and let G_n be the well-connected graph of Chapter 2. The main result of this chapter is the following.

Theorem 5.1 *Let A be an $n \times n$ matrix whose entries satisfy the relations $(L1)$, $(L2)$, and $(L3)$. Then there is a unique conductivity function γ on the graph G_n such that the response matrix $\Lambda_\gamma = A$.*

The set of all $n \times n$ matrices satisfying conditions $(L1)$, $(L2)$, and $(L3)$ will be called $L(n)$. The given matrix A in $L(n)$ can be joined to the matrix Λ_1 corresponding to $\gamma = 1$ by a path in $L(n)$. Theorem **??** will be proven by showing that each matrix on this path is the response matrix for a conductivity γ on G_n.

5.2 Some Matrix Algebra

In preparation for the proof of Theorem **??**, some non-standard, but easily established facts from matrix algebra are needed. Let M be any matrix. If every square sub-matrix of M (including M itself, if M happens to be square) has positive determinant, M is said to be *totally positive.* If every square sub-matrix of M (including M itself, if M is square) has non-negative determinant, M is said to be *totally non-negative.* Totally positive will sometimes be abbreviated TP, and totally non-negative will sometimes be abbreviated TNN.

More generally, let N be an array whose positions are a subset of the positions a rectangular matrix. If every square submatrix that can be formed from the entries of N has positive determinant, then N is said to be TP.

Recall that $M(i;j)$ is the *entry* in the (i,j) position of M; $M[i;j]$ is the matrix obtained from M by *deleting* the i^{th} row and the j^{th} column of M. If M is a $k \times k$ matrix, M^* will denote the array with $k^2 - 1$ entries formed from M by omitting the entry in the $(1,1)$ position. With the above terminology, to say that M^* is TP, means that every square sub-matrix of M which can be formed not using the entry $M(1;1)$ has positive determinant.

Lemma 5.2 *Suppose M is a $k \times k$ matrix for which M^* is totally positive. If $\det M > 0$, then M itself is totally positive.*

Proof: It is sufficient to show that every $(k-1) \times (k-1)$ sub-matrix of M has positive determinant. Let $M[h;j]$ be such a matrix. If either $i = 1$ or $j = 1$, $M[h;j]$ has positive determinant by assumption, so assume that $h \neq 1$ and $j \neq 1$. The six-term identity (Lemma **??**) shows that

$$\det M \det M[1,h;1,j] = \det M[1;1] \det M[h;j] - \det M[1;j] \det M[h;1]$$

Each of the determinants except det $M[h;j]$ is positive. Therefore det $M[h;j]$ must also be positive.

The expansion of det(M) of a $k \times k$ matrix by the cofactors of the first column is:

$$\det M = \sum_{i=1}^{i=k} (-1)^{i+1} M(i;1) \cdot \det M[i;1] \tag{5.1}$$

For each positive integer k, a function f_k is defined as follows. The function f_1 is defined to be identically 0. Let $k \geq 2$, and suppose M is a $k \times k$ matrix for which $\det M[1;1] \neq 0$. Then f_k is to be the function of the entries of M defined by the formula:

- $f_k(M) \cdot \det M[1;1] = \sum_{i=2}^{k} (-1)^i M(i;1) \cdot \det M[i;1]$

Observe that $f_k(M)$ is a function of the $k^2 - 1$ entries $M(i;j)$ for $(i,j) \neq (1,1)$. Since M^* is the array obtained from M by omitting the $(1,1)$-entry, this function can be written $f_k(M^*)$, which is well-defined if $\det M[1;1] \neq 0$. Let M be a $k \times k$ matrix such that M^* is TP. Lemma **??** implies that if the entry $M(1;1) > f_k(M^*)$, then M will be TP also.

5.3 Parametrizing Response Matrices

Suppose that A is an $n \times n$ matrix whose entries satisfy the relations $(L1)$, $(L2)$ and $(L3)$. If the values of $A(i;j)$ are given for all pairs (i,j) with $1 \leq i < j \leq n$, the entries below the diagonal are obtained from the symmetry relation: $A(i;j) = A(j;i)$. The diagonal entries are then obtained from the assumption that the row sums are 0. Thus the entries above the diagonal may be taken as parameters for A. The total number of parameters is $N = n(n-1)/2$, which is the same as the number of conductors in the well-connected graph G_n.

The next task is to parametrize the set $L(n)$, which is the set of $n \times n$ matrices satisfying $(L1)$, $(L2)$ and $(L3)$. For each $A \in L(n)$, for all circular pairs $(P;Q)$ with indices in the set $\{1,\ldots,n\}$, condition $(L3)$ says that the matrix $[-A(P;Q)]$ must have positive determinant. It is convenient to consider not A itself, but its negative $[-A]$, which means taking the negative of every entry of A. Thus condition $(L3)$ is replaced by $(L'3)$, where

$(L'3)$ For each circular pair $(P;Q) = (p_1,\ldots,p_k;q_1,\ldots,q_k)$,

$$\det A(P;Q) > 0$$

- $L'(n) = \{n \times n$ matrices A, satisfying $(L1)$, $(L2)$ and $(L'3)\}$.

A parametrization of the set $L'(n)$, will give a parametrization of $L(n)$.

For each fixed positive integer $n \geq 3$, let $\mathcal{P}$ (which depends on n) be the set of pairs of integers $\{(i,j)\}$ such that $1 \leq i < j \leq n$. The cardinality of $\mathcal{P}$ is $N = n(n-1)/2$. The set $\mathcal{P}$ will be ordered as follows. For each $(i,j) \in \mathcal{P}$, let $d(i,j)$ be the lesser of the two numbers $j-i$ and $n+i-j$. Thus if the vertices $v_1, v_2, \ldots, v_n$ are equally spaced on the boundary circle C, $d(i,j)$ is the shortest distance from v_i to v_j, measured either counterclockwise or clockwise around C, counting adjacent vertices as distance 1 apart. Let $m = [\frac{n}{2}]$. The set $\mathcal{P}$ is the disjoint union of the sets $\mathcal{P}_1, \mathcal{P}_j, \ldots, \mathcal{P}_m$, where $\mathcal{P}_k$ is the set of pairs (i,j) such that $d(i,j) = m+1-k$. The requirement on the ordering of $\mathcal{P}$ is that

- $\mathcal{P}$ may be ordered in any way such that $\mathcal{P}_1 < \mathcal{P}_2 < \ldots < \mathcal{P}_m$

That is, if $d(i,j) > d(i',j')$, then (i,j) is to precede (i',j') in the ordering. The pairs that are distance m occur first, then those that are distance $m-1$ apart, and finally the (i,j) that are adjacent.

Assume that a fixed choice of ordering of $\mathcal{P}$ is made, as above. This ordering gives a 1-1 correspondence between the set $\{1,\ldots,N\}$ and $\mathcal{P}$. For each $1 \leq a \leq N$, we must identify the largest square sub-matrix of the form $A(P;Q)$ where $(P;Q)$ is a circular pair, such that the a-th parameter position of A is in the upper left corner of $A(P;Q)$, and every other position of $A(P;Q)$ precedes a in the ordering.

Remark 5.1 The a-th parameter position will be in the *upper righthand* corner of a submatrix $A(P;R)$ of A where the indices of both P and R occur in the natural order (possibly after a circular shift). The indices of Q are in reverse order from the indices of R which makes $(P;Q)$ a pair in circular order. The only reason for doing this is that the signs are more manageable for $A(P;Q)$ than for $A(P;R)$.

The situation is slightly different depending on whether n is an odd integer or an even integer. Only the case of n odd will be considered here. The details of the even case are left to the reader.

Parametrizing $L(n)$ for n odd

Let m be a fixed positive integer, and $n = 2m+1$. For each $(i,j) \in \mathcal{P}$ let $k = m+1-d(i,j)$. A circular pair $(P;Q) = (p_1,\ldots,p_k;q_1,\ldots,q_k)$ of indices is defined as follows.

(1) If $j-i > m$, then $k = j-i-m$. $(P;Q)$ is defined by:

$$\begin{aligned} P &= (i, i+1, \ldots, i+k-1) \\ Q &= (j, j-1, \ldots, j-k+1) \end{aligned}$$

(2) If $j-i \le m$, then $k = m+1-j+i$. $(P;Q)$ is defined by:

$$\begin{aligned} P &= (j, j+1, \ldots, j+k-1) \\ Q &= (i, i-1, \ldots, i-k+1) \end{aligned}$$

In case (2), the matrix $A(P;Q)$ will be below the main diagonal of A. By a circular shift of the indices, $A(P;Q)$ will be above the main diagonal.

For $1 \le a \le N$, let (i,j) be the pair in $\mathcal{P}$ which corresponds to a in the ordering of $\mathcal{P}$, and let $A(P;Q)$ be the $k \times k$ matrix obtained above. Then this $A(P;Q)$ is the largest square sub-matrix such that the a-th parameter position of A is in the upper left corner of $A(P;Q)$, and every other position of $A(P;Q)$ precedes a in the ordering.

For $1 \le a \le 2m+1$, the matrix $A(P;Q)$ is the 1×1 matrix which is the entry in the position of A corresponding to a, and $M(a)$ is this entry. For $a > 2m+1$, write $a = a' + (k-1)n$ where $1 \le a' \le n$. Then the parameter position (i,j) corresponding to a is in the set $\mathcal{P}_k$, and $M(a)$ is a $k \times k$ sub-matrix $A(P;Q)$.

Example 5.1 Figure **??** shows an ordering of the parameter entries for a 7 × 7 matrix in A.

(1) Let $n = 7$, and suppose the ordering of the parameters is that given in Figure **??**. The $(1,6)$ position of A is numbered 13 in the ordering. For this entry, the circular pair is $P = (1,2)$ and $Q = (6,5)$, and

$$A(P;Q) = \begin{bmatrix} \Lambda(1,6) & \Lambda(1,5) \\ \Lambda(2,6) & \Lambda(2,5) \end{bmatrix}$$

$$\begin{bmatrix} * & 15 & 8 & 1 & 5 & 13 & 21 \\ & * & 16 & 9 & 2 & 6 & 14 \\ & & * & 17 & 10 & 3 & 7 \\ & & & * & 18 & 11 & 4 \\ & & & & * & 19 & 12 \\ & & & & & * & 20 \\ & & & & & & * \end{bmatrix}$$

Figure 5-1: Parameters positions for G_7

(2) For parameter number $a = 11$, the position is $(4, 6)$, the circular pair is $P = (6, 7)$ and $Q = (4, 3)$ and

$$A(P; Q) = \begin{bmatrix} \Lambda(6,4) & \Lambda(6,3) \\ \Lambda(7,4) & \Lambda(7,3) \end{bmatrix}$$

(3) For $a = 15$, the circular pair is $P = (2, 3, 4)$ and $Q = (1, 7, 6)$

$$A(P; Q) = \begin{bmatrix} \Lambda(2,1) & \Lambda(2,7) & \Lambda(2,6) \\ \Lambda(3,1) & \Lambda(3,7) & \Lambda(3,6) \\ \Lambda(4,1) & \Lambda(4,7) & \Lambda(4,6) \end{bmatrix}$$

For each $1 \leq a \leq N$, x_a stands for the value in the a-th parameter position of A. Let $M(a) = A(P; Q)$ where $A(P; Q)$ is the matrix for which the parameter position corresponding to a is the upper left corner, as described above. The other entries of $M(a)$ correspond to parameters x_b for $b < a$. For each integer $1 \leq a \leq N$, a function $F_a(x_1, \ldots, x_{a-1})$ is defined as follows. For $1 \leq a \leq n$, $F_a = 0$. Suppose inductively that F_b has been defined for $1 \leq b < a$.

- The domain of F_a is the set of all points $(x_1, x_2, \ldots, x_{a-1})$ in R^{a-1} such that for each $1 \leq b < a$, $x_b > F_b(x_1, x_2, \ldots, x_{b-1})$.

- $F_a(x_1, \ldots, x_{a-1})$ is defined to be $f_k(M(a)^*)$, where $M(a)$ is the $k \times k$ matrix obtained above, and $M(a)^*$ is the array with $k^2 - 1$ entries obtained from $M(a)$ by omitting the $(1, 1)$ entry, as defined in Section **??**.

Inductively, Lemma **??** shows that $F_a(x_1, \ldots, x_{a-1})$ is well-defined for each $a = 1, \ldots, N$, and, if $x_a > F_a(x_1, \ldots, x_{a-1})$, then $\det M(a) > 0$. Let S be the set of parameter values $x_1, x_2, \ldots, x_N$ such that for each $1 \leq a \leq N$, $x_a > F_a(x_1, x_2, \ldots, x_{a-1})$. An $n \times n$ matrix with relations $(L1)$ and $(L2)$ will also satisfy $(L'3)$ if and only if its parameter values $\{x_a\}$ lie in the set S. The set S is homeomorphic to $(R^+)^N$, as shown by the following Lemma.

Lemma 5.3 *Suppose f is a continuous function from D to R^+, where D is homeomorphic to $(R^+)^k$. Let $E \subseteq R^{k+1}$ be the set $E = \{x, y\}$ such that $y > f(x)$. Then E is homeomorphic to $(R^+)^{k+1}$.*

The homeomorphism θ is given explicitly by

$$\theta(x, y) = (x, y - f(x))$$

For each A in $L'(n)$, the matrix $[-A]$ is in $L(n)$. Therefore, $L(n)$ is also homeomorphic to $(R^+)^N$.

5.4 Principal Flow Paths

Suppose $\Gamma = (G_n, \gamma)$ is a resistor network whose underlying graph is the graph G_n constructed in Chapter **??**. The boundary vertices are $v_1, \ldots, v_n$, where as always $v_0 = v_n$. A family of γ-harmonic functions on Γ, will be constructed as in Section **??**. The indexing is slightly different for each of the congruence classes of n mod 4, so for definiteness, let $n = 4m + 1$. According to Lemma **??**, boundary values and boundary currents for a γ-harmonic function u (with boundary current ϕ) may be specified as follows.

$$\begin{aligned}
&u(v_0) = 1 \\
&u(v_j) = 0 \quad \text{for} \quad 2m + 1 \leq j \leq 4m \\
&\phi(v_j) = (-1)^j \quad \text{for} \quad 2m + 1 \leq j \leq 4m
\end{aligned}$$

Lemma **??** shows that there is a unique γ-harmonic function u with this boundary data. Harmonic continuation shows that u will be non-zero and alternate in sign for the boundary nodes v_i for $1 \leq i \leq 2m$.

$$(-1)^i u(v_i) > 0 \quad \text{for} \quad 1 \leq i \leq 2m$$

The direction of current flow along every edge in Γ can be determined as follows. Let $P = (v_1, v_2, \ldots, v_{2m})$ and $Q = (v_{4m}, v_{4m-1}, \ldots, v_{2m+1})$. Then $(P; Q)$ is a circular pair of boundary nodes of Γ, and there is a $2m$-connection α from P to Q through Γ, as described in Chapter 2. Specifically, v_j is connected to v_{n-j} by the path

$$v_j = (m+1, j) \xrightarrow{\eta} (m+1-j, j) \xrightarrow{\beta}$$
$$(m+1-j, n-j) \xrightarrow{\eta} (m+1, n-j) = v_{n-j}$$

Here η is the radial path in or out along the ray. For $1 \leq j \leq m$, β is the counterclockwise path along the circle, and for $m+1 \leq j \leq 2m$, β is the clockwise path along the circle. The process of harmonic continuation shows that

(1) For odd values of j, the values of u are strictly increasing along α_j. $u(v_j) < 0$ and $u(v_{4m+1-j}) = 0$.

(2) For even values of j, the values of u are decreasing along α_j from v_j to v_{4m+1-j}. $u(v_j) > 0$ and $u(v_{4m+1-j}) = 0$.

These paths α_j for $1 \leq j \leq 2m$ will be called *principal flow paths* for f. Every interior edge not on one of these principal flow paths joins a node on one principal flow path where u has positive sign to a node on another principal flow path where u has negative sign. These edges are said to be *transverse* to the principal flow paths. The direction of the current is known for every conductor in Γ.

Example 5.2 The principal flow paths for the function $u = u_0$ on the circular network G_7 are illustrated in Figure **??**. The edges of the graph along the principal flow paths are indicated by the solid lines, with the arrows indicating the direction of flow (decreasing potential). The edges transverse to the principal flow paths are indicated by the dotted lines. The boundary currents are indicated by the symbols (± 1).

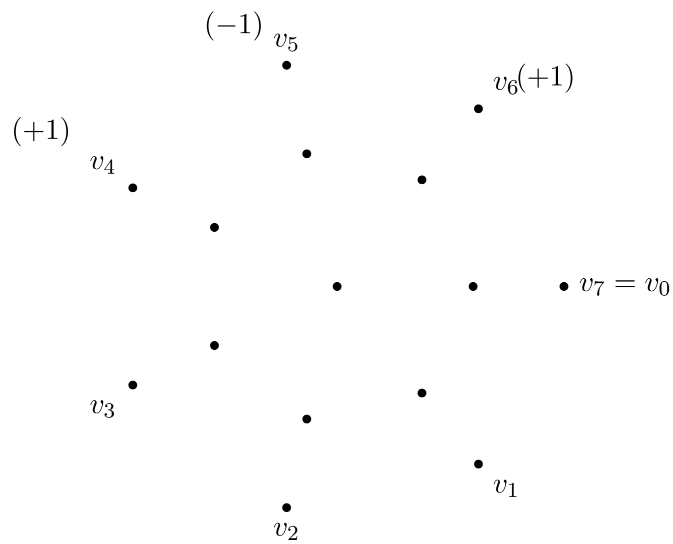


Figure 5-2:

Proposition 5.4 *In this situation,*

(1) If pq is any edge along a principal flow path, then $|c(pq)| > 1$.

(2) Suppose there is a bound B, such that for all $1 \leq j \leq n$, $\gamma(v_j w_j) < B$. Then for any interior edge transverse to the principal flow paths,

$$|u(p) - u(q)| > \frac{2}{B}$$

Proof: Suppose pq is an edge along a principal flow path α_j. (Assume j is even; the proof is similar for j odd.) Suppose

$$\alpha_j = r_0 r_1 \ldots r_{h-1} r_h r_{h+1} \ldots r_k$$

where $r_0 = v_j$ and $r_k = v_{n-j}$. Suppose the neighbors of r_h not on α_j are s_h and t_h. Then

$$c(r_{h+1}, r_h) + c(s_h, r_h) + c(t_h, r_h) = c(r_h, r_{h-1})$$

Each of these currents on the left hand side is ≥ 0. By induction along α_j, $c(r_{h+1}, r_h) \geq 1$. Therefore $c(r_h, r_{h-1}) \geq 1$.

(2) For each boundary node v_j, let w_j be its interior neighbor. Then for $2m+1 \leq j \leq 4m$, $|u(w_j)| \geq \dfrac{1}{B}$. Therefore for any edge pq with one endpoint on a principal flow path of positive sign and the other endpoint on a principal flow path of negative sign, $|u(p) - u(q)| > \dfrac{2}{B}$.

For any edge in G_n, there is a pattern of boundary data (obtained by a suitable rotation of Figure **??**) that places the chosen edge along a principal flow path. Similarly, for any edge in G_n, there is a pattern of boundary data that places the chosen edge transverse to the principal flow paths.

For each value of k with $0 \leq k \leq 4m+1$, there is a function u_k similar to the function $u = u_0$, obtained by rotating the graph. The current due to u_k is ϕ_k. The boundary conditions for u_k will be

$$\begin{aligned} u_k(v_k) &= 1 \\ u_k(v_{j+k}) &= 0 \quad \text{for} \quad 2m+1 \leq j \leq 4m \\ \phi_k(v_{j+k}) &= (-1)^j \quad \text{for} \quad 2m+1 \leq j \leq 4m \end{aligned}$$

For each $n \geq 3$, there are similar families of functions for the graph G_n.

5.5 Proof of Theorem ??

The proof of Theorem **??** will be given when n is an odd integer. The proof for n even is similar, but the indexing of the special functions is slightly different. Throughout the remainder of this section $n = 2m+1$ is a fixed odd integer, and $G = G_n$ is the well-connected critical graph of Chapter **??**. Lemma **??** implies that the set $L(n)$ is homeomorphic to $(R^+)^N$ and hence is path-connected. Let A be a matrix in $L(n)$. To prove Theorem **??** it is necessary to show that there is a conductivity function γ on G such that the response matrix $\Lambda_\gamma = A$.

- $R(n)$ will denote the set of all $n \times n$ response matrices Λ_γ for some conductivity γ on G.

It follows from Theorem **??**, Lemma **??** and the open mapping theorem that $R(n)$ is an open subset of the set of $L(n)$. Let $A(t)$ for $0 \leq t \leq 1$, be a path in $L(n)$ joining $A(0)$ with $A(1)$, where $A(0)$ is the λ-matrix corresponding to $\gamma = 1$, and where $A(1) = A$ is the given λ-matrix. Each matrix along this path is in $R(n)$. Suppose the contrary. Since the set of t for which $A(t)$ is in $R(n)$ is open, there is a least value t_0 for which $A(t_0)$ is not in $R(n)$. For each $t < t_0$, let $\gamma(t)$ be the conductivity corresponding to $A(t)$. For each edge pq, let $\mu(pq)$ be zero, infinity, or a positive real number such that there is a sequence $\{t_1, t_2, \ldots, t_k, \ldots\}$ with $\lim_{k\to\infty} t_k = t_0$, and such that $\lim_{k\to\infty}\gamma(t_k)(pq) = \mu(pq)$. We will write $\gamma^{(k)}$ for $\gamma(t_k)$ and $A^{(k)}$ for $A(t_k)$.

For each $k = 1, 2, \ldots$, let $\Gamma^{(k)} = (G, \gamma^{(k)})$ be the resistor network which has conductivity $\gamma^{(k)}$ on the graph G. We will make use of the principal flow paths described in Section **??**. Let $Q = (v_1, v_2, \ldots, v_m)$ and $P = (v_{m+1}, \ldots, v_{2m})$. For each k, let $M^{(k)} = A^{(k)}(P; Q)$, and $\psi^{(k)} = A^{(k)}(P; 2m+1)$. Let $c = c(v_1), \ldots, c(v_m)$ be the vector of currents $c(v_j) = (-1)^{j+1}$ for $1 \le j \le m$. Let $y^{(k)}$ be the vector with components $y^{(k)}(v_i)$ for $1 \le j \le m$, which is the solution to the linear system.

$$M^{(k)}y^{(k)} + \psi^{(k)} = c$$

Let $x^{(k)}$ be the boundary potential given by

(1) $x^{(k)}(v_{2m+1}) = 1$

(2) $x^{(k)}(v_j) = y^{(k)}(v_i)$ for $1 \le j \le m$,

(3) $x^{(k)}(v_j) = 0$ for $m+1 \le j \le 2m$.

Let $u^{(k)}$ be the $\gamma^{(k)}$-harmonic function with boundary values $x^{(k)}(v_j)$ for $1 \le j \le 2m+1$. Thus $u^{(k)}$ is the $\gamma^{(k)}$-harmonic function on $\Gamma^{(k)}$ with boundary current $\phi^{(k)}$, for which

$$\begin{aligned} u^{(k)}(v_0) &= 1 \\ u^{(k)}(v_j) &= 0 \quad \text{for} \quad 1 \le j \le m \\ \phi^{(k)}(v_j) &= (-1)^{j+1} \quad \text{for} \quad 1 \le j \le m \end{aligned}$$

(Recall that, by convention, $v_0 = v_{2m+1}$.)

Lemma 5.5 *In this situation, there is an upper bound for the magnitudes of $\mid u^{(k)}(p) \mid$ for all k and all nodes p in G. There is also an upper bound for the magnitudes of the currents $\mid \gamma^{(k)}(pq)(u^{(k)}(p) - u^{(k)}(q)) \mid$ for all edges pq in G.*

The proof of Lemma **??** uses two easily proven facts from matrix algebra.

Lemma 5.6 *Let $B^{(k)}$ be a sequence of $n \times n$ matrices with $\lim_{k\to\infty} B^{(k)} = B$. Let $v^{(k)}$ be a sequence of vectors of bounded norms. Then the norms of $B^{(k)}v^{(k)}$ and the magnitudes of $< v^{(k)}\,, B^{(k)}v^{(k)} >$ are bounded.*

Lemma 5.7 *Let $B^{(k)}$ be a sequence of $n \times n$ matrices with $\lim_{k\to\infty} B^{(k)} = B$. Assume that B and each $B^{(k)}$ is nonsingular. Let $c^{(k)}$ be a sequence of vectors of bounded norms. For each $k = 1, 2, \ldots$, let $v^{(k)}$ be the vectors with $B^{(k)}v^{(k)} = c^{(k)}$. Then the norms of $v^{(k)}$ are bounded.*

Proof: of Lemma **??**. $\lim_{k\to\infty} A^{(k)} = A^{(0)}$ and each of these is a λ-matrix. It follows from Lemma **??** that for any fixed boundary potential ϕ, the magnitudes of $< \phi\,, A^{(k)}(\phi) >$ are bounded. Also, each of the submatrices $A^{(k)}(P;Q)$ is non-singular. Lemma **??** shows that there is an upper bound for the values of $\mid x^{(k)}(v_j) \mid$ for all boundary nodes v_j and all k. By the maximum principle (Theorem **??**), this is also an upper bound for $\mid u^{(k)}(p) \mid$ for all k and all nodes p in G. Lemma **??** implies that there is an upper bound for the boundary currents for all k. By Theorem **??**, There is also an upper bound for the current along any edge in G.

(i) Assume that for some edge $e = pq$, $\mu(pq) = 0$. Whether radial or circular, by a rotation of the figure, assume that pq lies along a principal flow path α_j as in Figure **??**. Let $\gamma^{(k)}(pq) = \varepsilon_k$, where $\lim_{k\to\infty} \varepsilon_k = 0$. Let the γ-harmonic functions $u^{(k)}$ be as in Section **??**. By Proposition **??**, the current across edge $e = pq$ is at least 1, so

$$u^{(k)}(p) - u^{(k)}(q) \geq 1/\varepsilon_k$$

This would imply that $\lim_{k\to\infty} u^{(k)}(p) = \infty$, contradicting Lemma **??**.

(ii) Next assume that $\mu(ab) = \infty$ for some boundary spike ab. By a rotation of the graph G assume that $\mu(v_0w_0) = \infty$. Refer to Figure **??**.

For each positive integer k, let $u^{(k)}$ be the $\gamma^{(k)}$-harmonic function on G as in Section **??**. Let $X_k = \gamma^{(k)}(v_0w_0)$, where $\lim_{k\to\infty} X_k = \infty$. Then $u^{(k)}(v_0) = 1$, and $u^{(k)}(w_0) < 0$ This would imply that

$$\begin{aligned} c(v_0w_0) &= \gamma^{(k)}(v_0w_0)(v^{(k)}(v_0) - v^{(k)}(w_0)) \\ &> \gamma^{(k)}(v_0w_0)(1-0) \\ &> X_k \end{aligned}$$

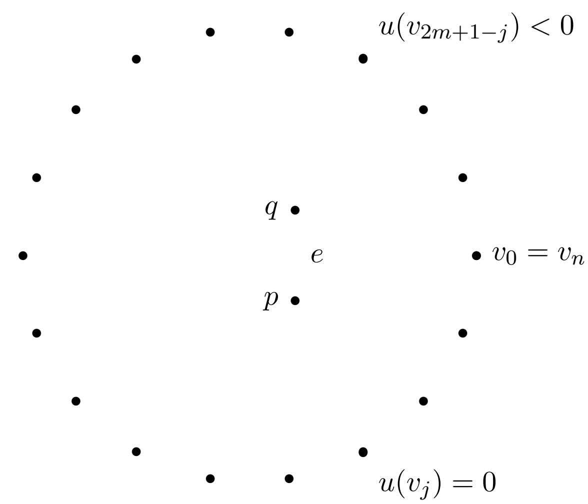


Figure 5-3: Edge along principal flow path

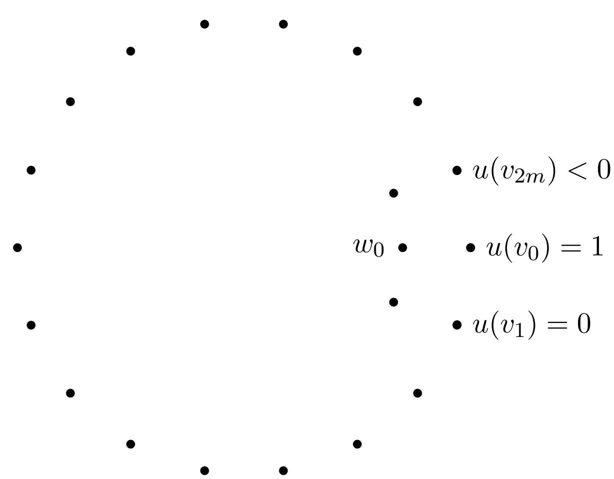


Figure 5-4: Spike

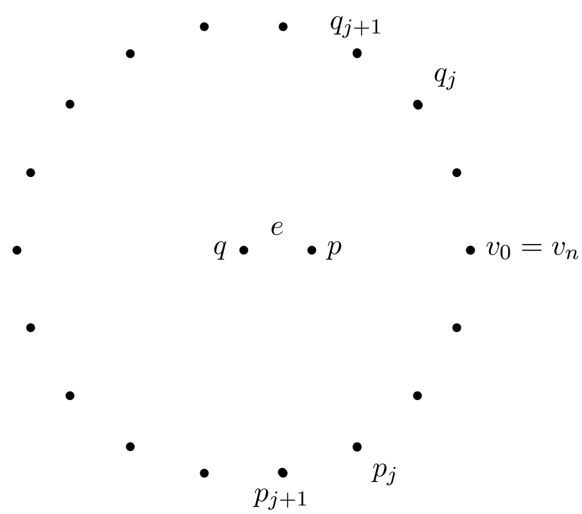


Figure 5-5: Edge transverse to principal flow path

which would contradict Lemma **??**.

From (i) and (ii), we may assume there are bounds ϵ and B, such that $\varepsilon \leq \gamma^{(k)}(ab) \leq X$ for each boundary spike $v_j w_j$ and each $k \geq 0$.

(iii) Assume that for some interior edge $e = pq$, $\mu(pq) = \infty$. Whether radial or circular, by a rotation of the figure, the edge pq may be assumed to be transverse to the principal current flow, as in Figure **??**. By Proposition **??**, this would give a current through pq which is

$$\gamma^{(k)}(pq)(u^{(k)}(q) - u^{(k)}(p)) \geq 2X^{(k)}/B$$

This has limit ∞, which contradicts Lemma **??**.

Thus any $n \times n$ matrix A in $L(n)$ has been shown to be of the form $A = \Lambda_\gamma$. This completes the proof of Theorem **??**.

Chapter 6

Adjoining Edges

Let $\Gamma = (G, \gamma)$ be a circular planar resistor network with n boundary nodes. There are three ways to adjoin an edge to Γ. The effect of each adjunction on the matrix Λ will be described.

6.1 Adjoining a Boundary Edge

Suppose p and q are two adjacent boundary nodes of a circular planar graph G. If a new edge joining p to q is added to the edge set of G, the new graph is again a circular planar graph with n boundary nodes. This process is called *adjoining a boundary edge* and the new graph is called $\mathcal{T}(G)$, where the two boundary nodes must be made clear from the context. If a boundary edge pq is adjoined to a circular planar resistor network $\Gamma = (G, \gamma)$, with $\gamma(pq) = \xi$, the resulting resistor network is denoted $\mathcal{T}(\Gamma)$, or $\mathcal{T}_\xi(\Gamma)$ when it is necessary to indicate the value of the adjoined conductor. Figure **??**a shows a circular planar graph G with 5 boundary nodes and 7 edges. Figure **??**b shows the graph $\mathcal{T}(G)$ with 8 edges which results when an edge from v_1 to v_2 is adjoined to this graph.

Suppose M is an $n \times n$ matrix, ξ is a real number, p and q are two adjacent indices from the index set for M. A new matrix $T_\xi(M)$ is defined

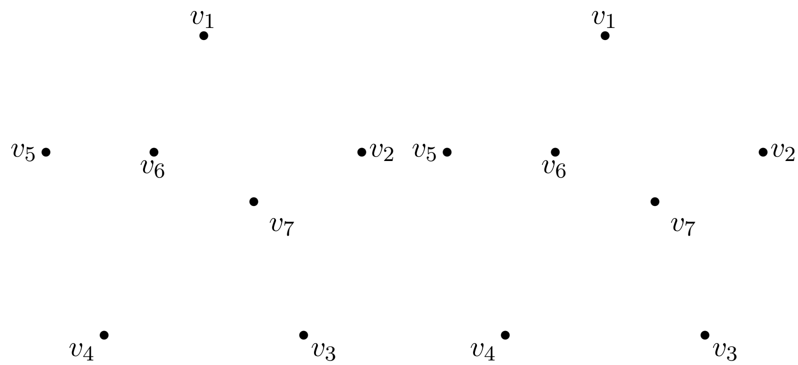


Figure 6-1: Adjoining a boundary edge v_1v_2

as follows.

$$\begin{aligned}
T_\xi(M)(p;p) &= M(p;p)+\xi \\
T_\xi(M)(q;q) &= M(q;q)+\xi \\
T_\xi(M)(p;q) &= M(p;q)-\xi \\
T_\xi(M)(q;p) &= M(q;p)-\xi \\
T_\xi(M)(i;j) &= M(i;j) \quad \text{otherwise}
\end{aligned}$$

Clearly, $T_{-\xi} \circ T_\xi =$ identity. It follows immediately from the definition of the Kirchhoff matrix that

$$K(\mathcal{T}_\xi(\Gamma)) = T_\xi(K(\Gamma))$$

Suppose $\Gamma = (G, \gamma)$ is a resistor network, and a pair of adjacent boundary nodes p and q, and a real number and ξ are given. From Theorem **??**, it follows that

$$\Lambda(\mathcal{T}_\xi(\Gamma)) = T_\xi(\Lambda(\Gamma))$$
$$\Lambda(\Gamma) = T_{-\xi}(\Lambda(\mathcal{T}_\xi(\Gamma)))$$

Thus $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathcal{T}_\xi(\Gamma))$, and $\Lambda(\mathcal{T}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$.

Observation 6.1 Suppose pq is a boundary edge with $\gamma(pq) = \xi$, If G' is the graph with edge pq deleted, then the response matrix Λ' for G' is expressed in terms of Λ as

$$\Lambda' = T_{-\xi}(\Lambda) \tag{6.1}$$

Example 6.1 Suppose the conductances of the edges of the graph of Figure **??**a are: $\gamma(v_1v_6) = 18$; $\gamma(v_2v_7) = 1$; $\gamma(v_3v_7) = 1$; $\gamma(v_4v_7) = 3$; $\gamma(v_4v_5) = 1$; $\gamma(v_5v_6) = 12$; $\gamma(v_6v_7) = 6$. The Kirchhoff matrix for this network is

$$K(\Gamma) = \begin{bmatrix} 18 & 0 & 0 & 0 & 0 & -18 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 & -1 & 0 & -3 \\ 0 & 0 & 0 & -1 & 13 & 12 & 0 \\ -18 & 0 & 0 & 0 & -12 & 36 & -6 \\ 0 & -1 & -1 & -3 & 0 & -6 & 11 \end{bmatrix}$$

The response matrix is the Schur complement of the 2 by 2 matrix in the lower right corner in K. The result is the matrix Λ:

$$\Lambda = \begin{bmatrix} 8.1 & -.3 & -.3 & -.9 & -6.6 \\ -.3 & .9 & -.1 & -.3 & -.2 \\ -.3 & -.1 & .9 & -.3 & -.2 \\ -.9 & -.3 & -.3 & 3.1 & -1.6 \\ -6.6 & -.2 & -.2 & -1.6 & 8.6 \end{bmatrix}$$

After adjoining an edge joining v_1 to v_2 with conductance $\gamma(v_1v_2) = 10$, the response matrix for the network $\mathcal{T}_{10}(\Gamma)$ is

$$\mathcal{T}_{10}(\Lambda) = \begin{bmatrix} 18.1 & -10.3 & -.3 & -.9 & -6.6 \\ -10.3 & 10.9 & -.1 & -.3 & -.2 \\ -.3 & -.1 & .9 & -.3 & -.2 \\ -.9 & -.3 & -.3 & 3.1 & -1.6 \\ -6.6 & -.2 & -.2 & -1.6 & 8.6 \end{bmatrix}$$

Let $P = (v_2, v_3)$ and $Q = (v_1, v_5)$. There is no 2-connection from P to Q through Γ, which corresponds to the fact that

$$\det \Lambda(2, 3; 1, 5) = \begin{bmatrix} -.3 & -.2 \\ -.3 & -.2 \end{bmatrix} = 0$$

After adjoining the edge v_1v_2,

$$\det T(\Lambda)(2,3;1,5) = \begin{bmatrix} -10.3 & -.2 \\ -.3 & -.2 \end{bmatrix} \neq 0$$

This non-zero determinant shows that there now is a 2-connection from P to Q through $\mathcal{T}(\Gamma)$. The paths of the connection are

$$\begin{array}{rcl} v_2 & \to & v_1 \\ v_3 & \to & v_7 \to v_6 \to v_5 \end{array}$$

There is no 2-connection from (v_2, v_3) to (v_4, v_5) through either Γ or through $\mathcal{T}(\Gamma)$. In the two cases,

$$T(\Lambda)(2,3;4,5) = \Lambda(2,3;4,5) = \begin{bmatrix} -.3 & -.2 \\ -.3 & -.2 \end{bmatrix}$$

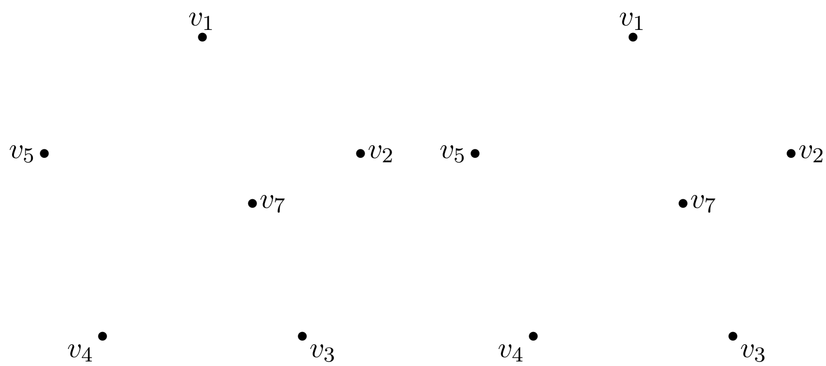
In each case, the determinant is 0.

Example 6.2 The indices 5 and 1 are adjacent in the circular ordering $\{5,1,2,3,4\}$. For $\xi = 6$, the operation T_{-6} on the response matrix Λ of Figure **??** produces the matrix $T_{-6}(\Lambda) = A$, where

$$A = \begin{bmatrix} 2.1 & -.3 & -.3 & -.9 & -0.6 \\ -.3 & .9 & -.1 & -.3 & -.2 \\ -.3 & -.1 & .9 & -.3 & -.2 \\ -.9 & -.3 & -.3 & 3.1 & -1.6 \\ -0.6 & -.2 & -.2 & -1.6 & 2.6 \end{bmatrix}$$

This matrix A is the response matrix for the network $\Gamma' = (G'', \gamma'')$ of Figure **??**b. The network Γ'' can be obtained from Γ in two steps as follows. First perform a $Y - \triangle$ transformation with vertex at v_6 on Γ to give the graph G' of Figure **??**a. The edge v_1v_5 in G' has conductance 6. The calculation of the effect of a $Y - \triangle$ transformation on the Kirchhoff matrix was made in Chapter **??**, Example **??**. If "an edge of conductance -6" is adjoined to the graph G' of Figure **??**a, the result would be that $\gamma(v_1, v_5) = 0$, indicated by the dotted line on the graph G'' of Figure **??**b.

Remark 6.1 With the indices v_1 and v_5, any value more negative than $\xi = -1$ would produce a matrix which violates property $P(2)$ of Chapter **??**.



(a) Graph G' (b) Graph G'' (v_1v_5 deleted)

Figure 6-2: Edge deletion

Notice that for the sub-matrix $\Lambda(1,2;4,5)$

$$\det \Lambda(1,2;4,5) = \begin{bmatrix} -.9 & -6.6 \\ -.3 & -.2 \end{bmatrix} \neq 0.$$

This corresponds to the fact that there is a 2-connection from (v_1, v_2) to (v_4, v_5). However, for the matrix $A = \Lambda(\Gamma'')$,

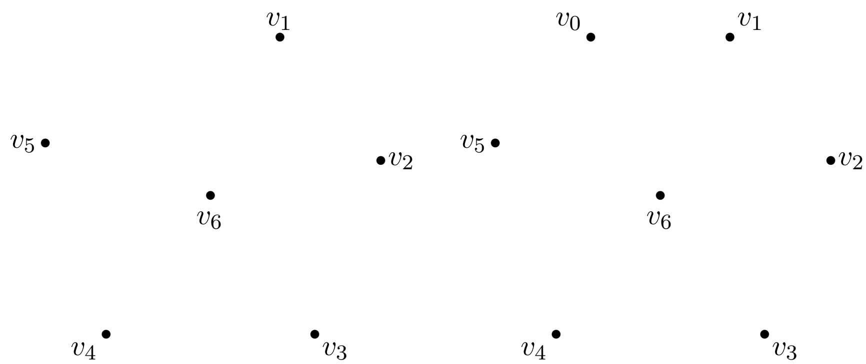
$$\det A(1,2;4,5) = \det \begin{bmatrix} -.9 & -0.6 \\ -.3 & -.2 \end{bmatrix} = 0$$

This shows that there is no 2-connection from (v_1, v_2) to (v_4, v_5) through G''.

6.2 Adjoining a Boundary Pendant

There are two closely related ways to adjoin a spike to a boundary node of Γ. One way increases the number of boundary nodes by one. The second way leaves the number of boundary nodes the same.

Let $\Gamma = (G, \gamma)$ be a circular planar resistor network and let p be a boundary node. By a cyclic re-labeling of the boundary nodes, we may assume that $p = v_1$. A new vertex v_0 is placed between v_n and v_1 on the



(a) Graph (b) Graph with pendant at v_1

Figure 6-3: Adjoining a boundary pendant at v_1

boundary circle C, and a new edge v_0v_1 is adjoined to Γ. The new graph is a circular planar graph with $n+1$ boundary nodes. We call this process *adjoining a boundary pendant*, and the resulting network is denoted $\mathcal{P}(\Gamma)$. If the new edge has conductance $\gamma(v_0v_1) = \xi$, and we need to refer to the value of the conductances, the resulting resistor network is denoted $\mathcal{P}_\xi(\Gamma)$.

Example 6.3 If a boundary pendant is adjoined to the graph of Figure **??**a at node v_1 the result is the graph of Figure **??**b.

Suppose M is an $n \times n$ matrix, written in block form:

$$M = \begin{bmatrix} M(1;1) & a \\ b & C \end{bmatrix}$$

If ξ a real number, let $P_\xi(M)$ be the $(n+1) \times (n+1)$ matrix, with indices $0 \le i \le n$ and $0 \le j \le n$. Then

$$P_\xi(M) = \begin{bmatrix} \xi & -\xi & 0 \\ -\xi & M(1;1)+\xi & a \\ 0 & b & C \end{bmatrix}$$

Suppose given the network $\Gamma = (G, \gamma)$ a boundary node p, and a real number ξ. By Theorem **??**,

$$\Lambda(\mathcal{P}_\xi(\Gamma)) = P_\xi(\Lambda(\Gamma))$$

Thus $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathcal{P}_\xi(\Gamma))$. Also, $\Lambda(\mathcal{P}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$.

Example 6.4 Suppose G is the graph of Figure **??**a. If a boundary pendant v_0v_1 is adjoined at v_1, the result is the graph $\mathcal{P}(G)$ of Figure **??**b. For the network $\Gamma = (G, \gamma)$ with the conductances $\gamma(v_1v_2) = 1$, $\gamma(v_1v_6) = 3$, $\gamma(v_2v_6) = 1$, $\gamma(v_3v_6) = 1$, $\gamma(v_4v_6) = 3$, $\gamma(v_4v_5) = 1$, $\gamma(v_5v_6) = 2$, the response matrix for Γ is

$$\Lambda(\Gamma) = \begin{bmatrix} 3.1 & -1.3 & -.3 & -.9 & -.6 \\ -1.3 & 1.9 & -.1 & -.3 & -.2 \\ -.3 & -.1 & .9 & -.3 & -.2 \\ -.9 & -.3 & -.3 & 3.1 & -1.6 \\ -.6 & -.2 & -.2 & -1.6 & 2.6 \end{bmatrix}$$

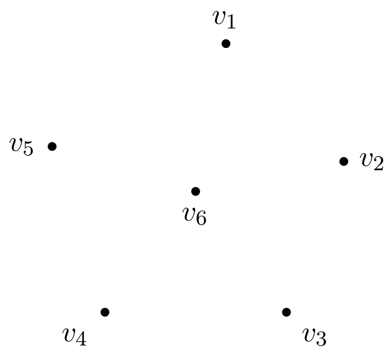
If an edge with conductance 1 is adjoined at v_1, the response matrix for the network $\mathcal{P}_1(\Gamma)$ of Figure **??**b is

$$\Lambda(\mathcal{P}_1(\Gamma)) = \begin{bmatrix} 1.0 & -1.0 & 0 & 0 & 0 & 0 \\ -1.0 & 4.1 & -1.3 & -.3 & -.9 & -.6 \\ 0 & -1.3 & 1.9 & -.1 & -.3 & -.2 \\ 0 & -.3 & -.1 & .9 & -.3 & -.2 \\ 0 & -.9 & -.3 & -.3 & 3.1 & -1.6 \\ 0 & -.6 & -.2 & -.2 & -1.6 & 2.6 \end{bmatrix}$$

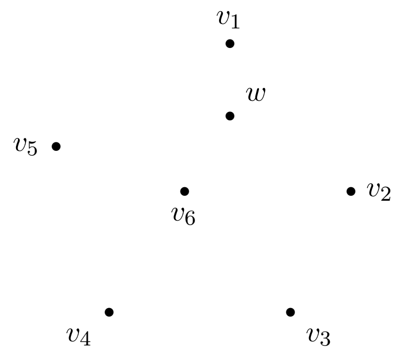
6.3 Adjoining a Boundary Spike

A spike may be adjoined at a boundary node, leaving the number of boundary nodes unchanged. By a cyclic re-labeling of the boundary nodes, assume that $p = v_1$. First a boundary pendant sv_1 is adjoined to Γ, and then v_1 is declared to be an interior node. The nodes are renumbered so that s takes the place of v_1 as the first boundary node. The new graph is a circular planar graph with n boundary nodes. This process is called *adjoining a boundary spike*. If a boundary spike sv_1 is adjoined to Γ, with $\gamma(sv_1) = \xi$, the resulting resistor network is denoted $\mathcal{S}_\xi(\Gamma)$.

Example 6.5 If a boundary pendant sv_1 is adjoined to the graph of Figure **??**a, at node v_1, and v_1 is then made interior and called w, the result is the graph of Figure **??**b. The node s is re-labeled as v_1.



(a) Graph



(b) Graph with spike at v_1

Figure 6-4: Adjoining a spike at v_1

Suppose M is an $n \times n$ matrix, written in block form:

$$M = \begin{bmatrix} M(1;1) & a \\ & \\ b & C \end{bmatrix}$$

For any real number ξ, the $(n+1)\times(n+1)$ matrix $P_\xi(M)$ has been defined in Section **??**. The indexing for the matrix $P_\xi(M)$ is $0 \le i \le n$ and $0 \le j \le n$. The $(1,1)$ entry of $P_\xi(M)$ is $\delta = M(1;1) + \xi$. If δ is not 0, we may take the Schur complement of this entry in $P_\xi(M)$, to obtain the matrix $S_\xi(M)$:

$$S_\xi(M) = P_\xi/[M(1;1) + \xi] = \begin{bmatrix} \xi - \frac{\xi^2}{\delta} & \frac{a\xi}{\delta} \\ & \\ \frac{b\xi}{\delta} & C - \frac{ba}{\delta} \end{bmatrix}$$

The indices of $S_\xi(M)$ are $\{1, 2, \ldots, n\}$, with index position 1 corresponding to the new node (now re-labeled v_1). A straightforward calculation shows that $S_{-\xi} \circ S_\xi$ = identity. From the definition of the Kirchhoff matrix in Chapter **??**, we have:

$$K(\mathcal{S}_\xi(\Gamma)) = K(\mathcal{P}_\xi(\Gamma))$$

Then $\Lambda(\mathcal{S}_\xi(\Gamma))$ is the Schur complement in $P_\xi(K(\Gamma))$ of the block corresponding to the enlarged set of interior nodes, which is $I \cup \{v_1\}$. From

Theorem **??**, it follows that

$$\Lambda(\mathcal{S}_\xi(\Gamma)) = S_\xi(\Lambda(\Gamma))$$

$$\Lambda(\Gamma) = S_{-\xi}(\Lambda(\mathcal{S}_\xi(\Gamma)))$$

Suppose given (Γ, γ) and the positive real number ξ. Then $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathcal{S}_\xi(\Gamma))$. Also $\Lambda(\mathcal{S}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$.

Observation 6.2 Suppose $\Gamma = (G, \gamma)$ is a resistor network with response matrix Λ, and pr is a boundary spike with $\gamma(pr) = \xi$. If G' is the graph with edge pr contracted, then the response matrix Λ' for G' is expressed in terms of Λ as

$$\Lambda' = S_{-\xi}(\Lambda) \tag{6.2}$$

Example 6.6 Suppose G is the graph of Figure **??**a. If an edge v_0v_1 of conductivity 1 is adjoined at v_1, and v_1 is made interior, the result is the graph $\mathcal{S}(G)$ of Figure **??**b. The response matrix for $\mathcal{S}(G)$ is the following.

$$\Lambda(\mathcal{S}(\Gamma)) = \begin{bmatrix} .7561 & -.3171 & -.0732 & -.2195 & -.1463 \\ -.3171 & 1.4878 & -.1951 & -.5854 & -.3902 \\ -.0732 & -.1951 & .7780 & -.3659 & -.2439 \\ -.2195 & -.5854 & -.3659 & 2.9024 & -1.7317 \\ -.1463 & -.3902 & -.2439 & -1.7317 & 2.5122 \end{bmatrix}$$

The response matrix $\Lambda(\mathcal{S}(G))$ may be calculated either by taking the Schur complement in the Kirchhoff matrix for $\mathcal{S}(G)$ of the 2 by 2 matrix corresponding to the two interior nodes, or by taking the Schur complement of the $(1,1)$ entry in $\mathcal{P}(\Gamma)$. The $(1,1)$ entry corresponds to the node which is made interior.

The boundary spike has been adjoined at v_1 for ease of notation. The constructions of the networks $\mathcal{P}_\xi(\Gamma)$ or $\mathcal{S}_\xi(\Gamma)$ may be made at any boundary node. The operations on matrices $P_\xi(M)$ or $S_\xi(M)$ may be made at any index. The only restriction on performing $S_\xi(M)$ at the index p is that the value $m_{p,p} + \xi$ must be non-zero. In each case, the location of the nodes (or indices) where the construction is made will be clear from the context.

6.4 Recovery of Conductances III

Suppose $\Gamma = (G, \gamma)$ is a circular planar resistor network, and G is critical as a graph. Let Λ_γ be the response matrix for Γ. The methods of Chapter **??** and this Chapter show that for all edges pq in G, the conductances $\gamma(pq)$ may be recovered from Λ_γ. The following is needed from Chapter **??**.

(1) Any critical circular planar graph G has at least one edge that is either a boundary edge or a boundary spike. In fact, Lemma **??** shows that it must have at least three such edges.

(2) Suppose that a boundary edge is deleted from, or a boundary spike is contracted in a critical graph G, resulting in a graph G'. Then G' is also critical.

(3) Suppose that a boundary edge is deleted from, or a boundary spike is contracted in a critical graph G. The endpoints of the geodesics in the medial graph can be used to find the connection that is broken.

Recall from Chapter **??** that the conductance of a boundary edge can be calculated by formula **??**. Suppose $\gamma(pq) = \xi$. Then $T_{-\xi}(\Lambda_\gamma)$ is the response matrix for the graph G' with boundary edge pq deleted. Similarly, the conductance of a boundary spike can be calculated by formula **??**. Suppose $\gamma(pr) = \xi$. Then $S_{-\xi}(\Lambda_\gamma)$ is the response matrix for the graph G' with boundary spike pr contracted. This is summarized in the following Theorem.

Theorem 6.1 *let $\Gamma = (G, \gamma)$ be a circular network whose underlying graph is critical. Then the conductances may be calculated from the response matrix Λ_γ. The map which sends the conductivity function γ to the response matrix Λ_γ is 1 - 1. Let γ and μ be two conductivities on G. If Λ_γ is sufficiently near to Λ_μ, then γ will be near to μ.*

Chapter 7

Characterization II

The goal of this Chapter is an (algebraic) characterization of the set of all response matrices for circular planar graphs. Let Ω_n be the set of n by n matrices A which satisfy the following three properties.

($P1$) A is symmetric; that is, $A(i;j) = A(j;i)$

($P2$) The sum of the entries in each row is 0.

($P3$) For each circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$,

$$(-1)^k \det A(P;Q) \geq 0$$

The main result is the following.

Theorem 7.1 *Let A be an n by n matrix whose entries satisfy the relations ($P1$), ($P2$), and ($P3$). Then there is a critical circular planar graph G, and a conductivity function γ on G such that the response matrix $\Lambda_\gamma = A$.*

Condition ($P3$) for Theorem **??** replaces condition ($L3$) for Theorem **??**. In this way, non-negative matrices play the same role for arbitrary response matrices on arbitrary (critical) graphs that positive matrices play for well-connected (critical) graphs.

7.1 Totally Non-negative Matrices

Though elementary, the matrix algebra in this Section is somewhat intricate. The important facts to be established are Lemmas **??** and **??**. This section retains the notations of Chapters **??** and **??**, except that the positions of the entries of a matrix A are indicated by subscripts. Thus, if $A = \{a_{i,j}\}$ is a matrix, $a_{i,j}$ is the entry at the (i, j) position. If $P = (p_1, \ldots, p_k)$ is an ordered subset of the rows of A, and $Q = (q_1, \ldots, q_m)$ is an ordered subset of the columns of A, then

- $A(P; Q)$ is the $k \times m$ sub-matrix of A formed from rows $p_1, \ldots, p_k$ of A, and columns $q_1, \ldots, q_m$ of A. Specifically,

$$A(P; Q)_{i,j} = a_{p_i,q_j}$$

- $A[P; Q]$ is the matrix obtained by *deleting* the rows for which the index is in P, and *deleting* the columns for which the index is in Q.

Thus $A[1; 1, 2]$ refers to the matrix obtained by deleting row 1, and columns 1 and 2 from A. $A[\quad; 1]$ refers to the matrix obtained by deleting the first column of A.

Following [**?**], a rectangular matrix A is called *totally non-negative* (abbreviation: TNN) if every square sub-matrix has determinant ≥ 0. The following facts about TNN matrices will be needed in Chapter **??**.

Lemma 7.2 *Suppose $A = \{a_{i,j}\}$ is an $m \times m$ matrix which is TNN and non-singular. Then every principal minor of A is non-singular.*

Proof: The proof is by induction on m. For $m = 1$, there is nothing to prove. Let $m > 1$. If the entry $a_{1,1} = 0$, and the first row of A is not all 0, and the first column of A is not all 0, there would be a submatrix of A of the form

$$\begin{bmatrix} 0 & a_{1,j} \\ a_{i,1} & a_{i,j} \end{bmatrix}$$

which would have negative determinant. Thus if $a_{1,1} = 0$, then either the entire first row or the entire first column of A would be 0, contradicting the assumption that A is non-singular. This shows that the entry $a_{1,1}$ must be positive. By the determinantal formula for Schur complements, the Schur complement $A/[a_{1,1}]$ is non-singular and TNN. Similarly $a_{m,m} > 0$, $A/[a_{m,m}]$ is non-singular and TNN. By the inductive assumption, every principal minor of $A/[a_{1,1}]$ is non-singular. Let $A(P; P)$ be a principal minor of A, where $P = (p_1, \ldots, p_k)$ is an ordered subset of the index set $(1, 2, \ldots, m)$. If

1 is in P, $A(P;P)/[a_{1,1}]$ is a principal minor of $A/[a_{1,1}]$ and hence is non-singular. Thus $\det A(P;P) \neq 0$, so $A(P;P)$ is non-singular. Similarly if m is in P, $A(P;P)$ is non-singular. Otherwise, P contains neither 1 nor m, and $k \leq m-2$. Let $Q = (1, p_1, \ldots, p_m)$. The $k+1 \times k+1$ matrix $A(Q;Q)$ is TNN and non-singular. $A(P;P)$ is a principal minor of $A(Q;Q)$, so is non-singular by induction.

Recall that if M is in Ω_n, and $(P;Q)$ is a circular pair of indices, then the matrix $-M(P;Q)$ is TNN. We need to see what happens to $M(P;Q)$ if we take the Schur complement in M of a diagonal entry $m_{h,h}$. If the indexing set for M is $\{1, \ldots, n\}$, it is convenient to regard the deleted set $(1, \ldots, \hat{h}, \ldots, n)$ as the indexing set for M'. Let $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ be a circular pair of sequences of indices for M'. Then $h \notin P \cup Q$. By interchanging P and Q if necessary, and by a cyclic re-ordering of the indices, we may assume that $1 \leq h < q_k$ in the circular order. Figure **??** shows an example (with $k = 4$, and $p_2 < h < p_3$) of the type of submatrix of M under consideration. The submatrix of Λ consists of the entries whose locations are marked with an *. The top row lists the column indices, and the left side lists the row indices that are used to form the submatrix. Specifically, a matrix $M(R;T)$ is formed as follows. Let $R = (r_1, r_2, \ldots, r_{k+1})$, be the set $P \cup h$ with the circular ordering, where r_s is the index h and $1 \leq s \leq k+1$. Let $T = \{t_1, t_2, \ldots, t_{k+1}\}$ where $t_1 = h$, and for each $2 \leq i \leq k+1$, $t_i = q_{k+2-i}$. The matrix $M(R;T)$ has the form of Figure **??**. The entry at position (i,j) of $M(R;T)$ is given as follows.

- row i of $M(R;T)$ is taken from row p_i of M if $1 \leq i < s$
- row s of $M(R;T)$ is taken from row h of M
- row i of $M(R;T)$ is taken from row p_{i-1} of M if $s < i \leq k+1$
- column 1 of $M(R;T)$ is taken from column h of M
- column j of $M(R;T)$ is taken from column q_{k+2-j} of M if $1 < j \leq k+1$.

The matrix of interest is $A = -M(R;T)$. The entry at the $(s,1)$ position of A is $-m_{h,h}$, where $m_{h,h}$ is the diagonal entry of M. In the example shown in Figures **??** and **??**, the entry $A_{3,1} = -m_{h,h}$.

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \cdots & h & \cdots & q_4 & q_3 & q_2 & q_1 & \cdot \end{array} \\
\begin{array}{r} . \\ . \\ p_1 \\ p_2 \\ h \\ p_3 \\ p_4 \\ . \\ . \end{array} &
\begin{bmatrix}
\cdots & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots & * & \cdots & * & * & * & * & \cdot \\
\cdots & * & \cdots & * & * & * & * & \cdot \\
\cdots & * & \cdots & * & * & * & * & \cdot \\
\cdots & * & \cdots & * & * & * & * & \cdot \\
\cdots & * & \cdots & * & * & * & * & \cdot \\
\cdots & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
\end{array}$$

Figure 7-1:

$$\begin{array}{cc}
 & \begin{array}{ccccc} h & q_1 & q_2 & q_3 & q_4 \end{array} \\
\begin{array}{r} p_1 \\ p_2 \\ h \\ p_4 \\ p_5 \end{array} &
\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{bmatrix}
\end{array}$$

Figure 7-2:

Lemma 7.3 *Suppose that $A = \{a_{i,j}\}$ is an $m \times m$ matrix. Assume for some index s with $1 \leq s \leq m$ that*

(i) $a_{s,1} < 0$

(ii) $A[\quad;1]$ is TNN.

(iii) $A(s+1,\ldots,m;1,\ldots,m)$ is TNN.

(iv) $A(1,\ldots,s-1;2,\ldots,m,1)$ is TNN.

Then

(1) $(-1)^s \det A \geq 0$.

(2) If it is further assumed that $\det A[s;1] > 0$, *then* $(-1)^s \det A > 0$.

Proof: By induction on m. The assertion of (1) for $m = 2$ is immediate. Suppose $m > 2$ and first consider the case $s = 1$, with $a_{1,1} < 0$. If all the cofactors of the entries in the first column are 0, then $\det A = 0$. If the only non-zero cofactor of an entry in the first column is $A[1;1]$, then

$$\det A = a_{1,1} \cdot \det A[1;1] < 0$$

Otherwise, suppose $\det A[t;1] > 0$ with $t > 1$. $A[1,t;1,2]$ is a principal minor of $A[t;1]$ which is assumed to be TNN, so $\det A[1,t;1,2] > 0$ by Lemma **??**. The six-term identity (Lemma **??**) gives

$$\det A \cdot \det A[1,t;1,2] = \det A[1;1] \cdot \det A[t;2] - \det A[1;2] \cdot \det A[t;1]$$

$\det A[1;2]$ and $\det A[t;1]$ are non-negative by assumption (iii). By the inductive assumption $\det A[t;2] \leq 0$. Hence $\det A \leq 0$.

The case $s = m$ is similar, by considering the matrix $A(1,\ldots,m;2,\ldots,m,1)$. The only negative entry is in the last column. Assumption (iv) is used in place of (iii).

This leaves the case when $1 < s < m$. First suppose the only non-zero cofactor of an entry in the first column is $A[s;1]$. In this case,

$$\det A = (-1)^{s+1} \cdot a_{s,1} \cdot \det A[s;1]$$

If more than one cofactor is non-zero, assume that $\det A[s;1] > 0$ and $\det A[t;1] > 0$ with $1 < s < t \leq m$. Then $A[1,t;1,2]$ is a principal minor of $A[t;1]$, so $\det A[1,t;1,2] > 0$ by Lemma **??**. The six-term identity gives

$$\det A \cdot \det A[1,t;1,2] = \det A[1;1] \cdot \det A[t;2] - \det A[1;2] \cdot \det A[t;1]$$

The factors $\det A[1;1]$ and $\det A[t;1]$ are non-negative. By the inductive assumption, $(-1)^s \det A[t;2] \geq 0$ and $(-1)^{s-1} \det A[1;2] \geq 0$. In every case, $(-1)^s \det A \geq 0$.

The proof of (2) is also by induction on m. For $m = 2$, the assertion is immediate. Suppose $m > 2$. If the only non-zero cofactor of an entry in the first column is $A[s;1]$, then

$$(-1)^s \det A = -a_{s,1} \cdot \det A[s;1] > 0$$

If more than one cofactor is non-zero, assume that $\det A[s;1] > 0$ and $\det A[t;1] > 0$ with $1 < s < t \leq m$. Then $\det A[1,s;1,2] > 0$ and $\det A[1,t;1,2] > 0$ by Lemma **??**. By the inductive assumption,
$(-1)^{s-1} \det A[1;2] > 0$, and equation (**??**) shows that $(-1)^s \det A > 0$.

Lemma 7.4 *Let M be a matrix in Ω_n and suppose that $m_{h,h}$ is a non-zero diagonal entry. Then the Schur complement $M' = M/[m_{h,h}]$ is in Ω_{n-1}. Furthermore, if $(P;Q)$ is a circular pair of indices neither of which includes the index h, for which $\det(-1)^k M(P;Q) > 0$, then $\det(-1)^k M'(P;Q) > 0$.*

Proof: Let $(P;Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ be a circular pair of indices for M'. Then $h \notin P \cup Q$. As in the discussion prior to Lemma **??**, $R = (r_1, r_2, \dots, r_{k+1})$ is taken to be the set $P \cup h$ with the circular ordering, where r_s is the index h, and where $1 \leq s \leq k+1$. $T = \{t_1, t_2, \dots, t_{k+1}\}$ is the set $t_1 = h$, and for each $1 \leq i \leq k$, $t_{i+1} = q_i$. Thus $1 \leq r_1 < \dots r_{k+1} < t_{k+1} < \dots t_1 \leq n$. The matrix $A = -M(R;T)$ has the form of Figure **??**, and satisfies conditions (i)-(iv) of Lemma **??**. Hence $(-1)^s \det A \geq 0$, so

$$(-1)^{s+1+k} \det M(R;T) \geq 0$$

The entry $m_{h,h}$ in M is in the $(s,1)$ position of A. Taking the Schur complement of $m_{h,h}$ in M, gives:

$$M'(P;Q) = M(R;T)/[m_{h,h}]$$

Thus, by Lemma **??**, $(-1)^k \det M'(P;Q) \geq 0$. If $(-1)^k \det M(P;Q) > 0$, then part (2) of Lemma **??** shows that $(-1)^{s+1+k} \det M(R;T) > 0$. Therefore $(-1)^k \det M'(P;Q) > 0$.

Notation: Let $P = (p_1, p_2, \ldots, p_k)$ be a sequence of distinct indices.

• If $p \in P$, then $P - p$ denotes the sequence obtained by deleting the index p from P.

• If $p \notin P$, then $p + P$ denotes the sequence $(p, p_1, \cdots, p_k)$, and $P + p$ denotes the sequence $(p_1, \cdots, p_k, p)$.

Lemma 7.5 *Suppose M is in Ω_n, p and q are adjacent indices, and $\xi > 0$. Let $T_\xi(M)$ be the matrix constructed in Section **??**. Then $T_\xi(M)$ is in Ω_n.*

Proof: The circular determinants in $M' = T_\xi(M)$ are the same as the circular determinants in M except for the ones which correspond to circular pairs $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots q_k)$ where $p = p_k$ and $q = q_k$, or $p = p_1$ and $q = q_1$. Each of these determinants has the form

$$\begin{aligned}
\det M'(P;Q) &= \det \begin{bmatrix} C & a \\ b & d-\xi \end{bmatrix} \\
&= \det \begin{bmatrix} C & a \\ b & d \end{bmatrix} - \xi \det(C) \\
&= \det M'(P;Q) - \xi \det M(P-p; Q-q)
\end{aligned}$$

Hence

$$\begin{aligned}
&(-1)^k \det M'(P;Q) = \\
&(-1)^k \det M(P;Q) - \xi(-1)^{k-1} \det M(P-p;Q-q) \geq 0
\end{aligned}$$

Observation 7.1 If either $(-1)^k \det M(P;Q) > 0$ or $(-1)^{k-1} \det M(P - p; Q - q) > 0$, then $(-1)^k \det M'(P;Q) > 0$. Otherwise $\det M'(P;Q) = 0$.

Lemma 7.6 *Suppose M is in Ω_n, and $\xi > 0$. Let $P_\xi(M)$ be the matrix constructed in Section **??**. Then $P_\xi(M)$ is in Ω_{n+1}.*

Proof: Let $M' = P_\xi(M)$, and let $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ be a circular pair of indices from the set $(0, 1, \ldots, n)$.

(1) If $0 \notin P \cup Q$, then $\det M'(P;Q) = \det M(P;Q)$.

(2) If 0 is an index in P and $1 \notin Q$, then $\det M'(P;Q) = 0$.

(3) If 0 is in P and 1 is in Q, then $0 = p_k$, $1 = q_k$, and

$$\det M'(P;Q) = -\xi \det M(P - p_k; Q - q_k)$$

.

(4) The situation is similar if 0 is in Q.

Lemma 7.7 *Suppose M is in Ω_n, and $\xi > 0$. Let $S_\xi(M)$ be the matrix constructed in Section* **??**. *Then $S_\xi(M)$ is in Ω_n.*

Proof: Let $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ be a circular pair of indices. Let p be the index where the spike is adjoined. By interchanging P and Q if necessary, and by a circular re-0 of the indices, we may assume that $1 \leq p < q_k$ in the circular order. We first note that $S_\xi(M)$ is the Schur complement in $P_\xi(M)$ of the entry at the (p,p) position. Hence $S_\xi(M)$ is in Ω_n. Also

(1) If p is in P, then the formula for $S_\xi(M)$, shows that

$$\det M'(P;Q) = \left(\frac{\xi}{\xi + m_{p,p}}\right) \det M(P;Q)$$

(2) Suppose that p is not in P and $(-1)^k \det M(P;Q) > 0$. Then by Lemma **??**, $(-1)^k S_\xi(M)(P;Q) > 0$.

7.2 Characterization of Response Matrices II

Lemma 7.8 *Suppose M is in Ω_n, with at least one circular determinant equal to 0. Let $\epsilon > 0$ be given. Then there is a matrix M' in Ω_n, with $\|M' - M\|_\infty < \epsilon$, and*

(1) $\det M'(P;Q) \neq 0$ *whenever* $\det M(P;Q) \neq 0$

(2) For at least one circular pair $(P;Q)$*,*

$$\det M(P;Q) = 0 \quad \text{and}$$
$$\det M'(P;Q) \neq 0.$$

Proof: Let $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ be a circular pair of indices for which $\det M(P;Q) = 0$, and has minimum order k, and among those, take a circular pair $(P;Q)$ for which $|q_k - p_k|$ is a minimum.

(1) If $q_k - p_k = 1$, let $M' = T_\xi(M)$, where the chosen indices are p_k and q_k. By Lemma **??**, $\det M'(P;Q) \neq 0$. Also by Lemma **??**, $\det M'(R;S) \neq 0$ whenever $(R;S)$ is a circular pair for which $\det M(R;S) \neq 0$. If ξ is sufficiently small, then $\|M' - M\|_\infty < \epsilon$.

(2) If $q_k - p_k > 1$, let $p = p_k + 1$ and $M' = S_\xi(M)$ where the chosen index is p. By Lemma **??**, $\det M'(R;S) \neq 0$ whenever $(R;S)$ is a circular pair for which $\det M(R;S) \neq 0$.

We need to show that $\det(-1)^k M'(P;Q) > 0$. For simplicity of notation, N will denote $P_\xi(M)$ where the spike is adjoined at index p' with $p_k < p' < p_k + 1 = p$. Then $M' = S_\xi(M)$ will be the Schur complement of the (p,p) entry in N.

$$\begin{array}{c|ccccc|}
 & p & q_k & \cdots & q_2 & q_1 \\
\hline
p_1 & * & * & \cdots & * & * \\
p_2 & * & * & \cdots & * & * \\
\cdots & . & . & \cdots & . & . \\
p_k & * & * & \cdots & * & * \\
p & * & * & \cdots & * & *
\end{array}$$

The six-term identity of Lemma **??** gives

$$\begin{aligned}
&\det N(P+p; Q+p) \cdot \det N(P-p_k; Q-q_k) \\
&\qquad = \det N(P - p_k + p; Q - q_k + p) \cdot \det N(P;Q) \\
&\qquad\qquad - \det N(P - p_k + p; Q) \cdot \det N(P; Q - q_k + p)
\end{aligned}$$

Using the assumption that $\det M(P;Q) = 0$ implies $\det N(P;Q) = 0$. then

$$\det N(P+p;Q+p) = -\frac{\det N(P-p_k+p;Q)\cdot \det N(P;Q-q_k+p)}{\det N(P-p_k;Q-q_k)}$$

Each of the factors on the right hand side is non-zero because of the assumption of the minimality of $(P;Q)$. Therefore $\det N(P+p;Q+p) \neq 0$. Taking the Schur complement of the (p,p) entry yields $\det M'(P;Q) \neq 0$. If ξ is taken sufficiently large, then $\|M' - M\|_\infty < \epsilon$.

Proof: of Theorem **??**. Recall from Section **??** the graph $G_n = (V, V_B, E)$, with n boundary nodes.

Lemma **??** implies that Ω_n is the closure of $L(n)$ in the space of $n \times n$ matrices. Thus for any M in Ω_n, there is a sequence of matrices M_i in $L(n)$ which converge to M. Theorem **??** shows that for each integer i, there is a conductivity γ_i on G_n with $M_i = \Lambda(G_n, \gamma_i)$. By taking a subsequence if necessary, assume for each edge e in E that $\lim_{i\to\infty} \gamma_i(e)$ is either 0, a finite non-zero value or ∞.

Let E_0 be the subset of E for which $\lim\limits_{i\to\infty} \gamma_i(e) = 0$.

Let E_1 be the subset of E for which $\lim\limits_{i\to\infty} \gamma_i(e) = \gamma(e)$ is a finite non-zero value.

Let E_∞ be the subset of E for which $\lim\limits_{i\to\infty} \gamma_i(e) = \infty$.

Let $\Gamma = (W, V_B, E_1)$ be the graph obtained from $G_n = (V, V_B, E)$ by deleting the edges of E_0 and contracting each edge of E_∞ to a point. The vertex set W for Γ is the set of equivalence classes of vertices in V, where $p \sim q$ if pq is in E_∞. Note that distinct boundary nodes in V_B cannot belong to the same equivalence class, because the M_i are bounded. Thus V_B may be considered as a subset of W. Each edge e in E_1 joins a pair of points of W, so the edge-set of Γ is E_1. The restrictions of γ_i and γ to E_1 give conductivities on Γ. We shall show that $M = \Lambda(\Gamma, \gamma)$.

Suppose f is a function defined on the boundary nodes of Γ. Let

$$Q(f) = \inf \sum_{e \in E_1} \gamma(e)(\Delta w(e))^2$$

The infimum is taken over all functions w defined on the nodes of Γ which agree with f on V_B, and $\Delta w(pq) = w(p) - w(q)$, This infimum is attained when $w = u$ is the potential function on the resistor network (Γ, γ), with boundary values f; that is when w is γ-harmonic at each interior node of Γ. Similarly, for each integer i, let

$$Q_i(f) = \inf \sum_{e \in E_1} \gamma_i(e)(\Delta w(e))^2$$

This infimum is attained when $w = u_i$ is the potential function on (Γ, γ_i) with boundary values f. Then $\lim_{i \to \infty} u_i = u$, because the γ_i and γ are conductivities (non-zero, and finite) on Γ, with $\lim_{i \to \infty} \gamma_i = \gamma$. Therefore $Q(f) = \lim_{i \to \infty} Q_i(f)$.

For each integer i, let

$$S_i(f) = \inf \sum_{e \in E} \gamma_i(e)(\Delta w(e))^2$$

where the infimum is taken over all functions w defined on the nodes of G_n which agree with f on V_B. This infimum is attained when $w = w_i$ is the potential function on the resistor network (G_n, γ_i), with boundary values f. The maximum principle implies that $| w_i(p) | \leq \max | f(p) |$. By taking a subsequence if necessary, assume that for each node p, $w_i(p)$ converges to a finite value $w(p)$. The assumption that the M_i converge to M guarantees that for each function f, the $S_i(f)$ are bounded. Thus for each edge $e = pq \in E_\infty$, we have $w(p) = w(q)$. Let

$$R_i(f) = \sum_{e \in E_1} \gamma_i(e)(\Delta w_i(e))^2$$

and

$$R(f) = \lim_{i \to \infty} R_i(f) = \sum_{e \in E_1} \gamma(e)(\Delta w(e))^2$$

Let $\mathcal{F}$ be the set of functions $v = \{v(p)\}$ defined for all nodes of G_n, which agree with f on V_B, and for which $v(p) = v(q)$ whenever $pq \in E_\infty$. Let

$$P_i(f) = \inf_{v\in\mathcal{F}} \sum_{e\in E} \gamma_i(e)(\Delta v(e))^2$$

Then

$$P_i(f) \geq S_i(f) \geq R_i(f)$$

and

$$Q_i(f) + \sum_{e\in E_0} \gamma_i(e)(\Delta u_i(e))^2 \geq P_i(f) \geq Q_i(f)$$

The maximum principle implies that the $\mid u_i(p) \mid$ are bounded by $\max \mid f(p) \mid$. For each edge $e \in E_0$, we have $\lim_{i\to\infty} \gamma_i(e) = 0$, so

$$Q(f) = \lim_{i\to\infty} Q_i(f) = \lim_{i\to\infty} P_i(f) \geq \lim_{i\to\infty} R_i(f) = R(f)$$

But $R(f) \geq Q(f)$, so $R(f) = Q(f)$. Thus

$$\lim_{i\to\infty} S_i(f) = Q(f) = \lim_{i\to\infty} < f, M_i(f) > \ = \ < f, M(f) >$$

If G is a circular planar graph with n boundary nodes, let $\pi(G)$ be the set of connections $P \leftrightarrow Q$ through G. If $\pi = \pi(G)$ for some circular planar graph with n boundary nodes, let

- $\Omega_n(\pi) = \{n$ by n matrices M which satisfy $(-1)^k \det M(P;Q) > 0$ for each circular pair $(P;Q) \in \pi\}$.

Corollary 7.9 *Suppose $\Gamma = (G, \gamma)$ is a critical circular planar resistor network with N edges and $\pi = \pi(\Gamma)$. Then the map L which sends γ to Λ_γ is a diffeomorphism of $(R^+)^N$ onto $\Omega(\pi)$.*

Ω_n is the disjoint union of the sets $\Omega_n(\pi)$. Let $\omega = \pi(G_n)$, where G_n is the well-connected graph of Chapter **??**. Then $\Omega(\omega) = L(n)$. For more details, see also [**?**].

Chapter 8

Medial Graphs

8.1 Constructing the Medial Graph

Suppose G is a circular planar graph, with n boundary nodes; G is embedded in the plane so that the boundary nodes $v_1, v_2, \ldots, v_n$ are in clockwise order around a circle C and the rest of G is in the disc D which is the interior of C. The construction of the medial graph $\mathcal{M}$ is similar to that in [**?**] (p. 241), and depends on the embedding of G in the disc. For each edge e of G, let m_e be its midpoint. Next $2n$ points $t_1, t_2, \ldots t_{2n}$ are placed on C so that:

$$t_1 < v_1 < t_2 < t_3 < v_2 < \ldots < t_{2n-1} < v_n < t_{2n} < t_1$$

(1) The vertices of $\mathcal{M}$ consist of the points m_e for all edges e in G, and the points t_i for $i = 1, 2, \ldots, 2n$.

(2) If e and f are edges in G, with a common vertex in G, and which are incident to the same face in G, the line $m_e m_f$ joining the midpoints m_e and m_f is to be an edge in $\mathcal{M}$, and is called the *median*. For each point t_j on the boundary circle, there is one edge as follows. The point t_{2i} is joined by an edge to m_e where e is the edge of the form $e = v_i w$ which comes first after the line $v_i t_{2i}$ in clockwise order around v_i; the point t_{2i-1} is joined by an edge to m_f where f is the edge of the form $f = v_i w$ which comes first after the line $v_i t_{2i-1}$ in counter-clockwise order around v_i.

The vertices of the form m_e of $\mathcal{M}$ are in the interior of D and are 4-valent. The vertices of the form t_i on the bounding circle C are 1-valent.

Example 8.1 Figure **??**a shows a graph G with four boundary nodes v_1, v_2, v_3, v_4, two interior nodes v_5, v_6 and six edges. The edges of G (indicated by solid lines) are v_1v_2, v_2v_6, v_3v_6, v_4v_5, v_1v_5 and v_5v_6. The midpoint of each edge v_iv_j, is denoted by $m_{i,j}$. The points $t_1, t_2, \ldots, t_8$ are placed on the boundary circle C. The medians are the line segments indicated by the dotted lines. Figure **??**b shows a (topologically equivalent) version of the medial graph $\mathcal{M}$ without the underlying graph G; the corners of the medial lines have been smoothed, and the geodesics (defined in the paragraph below) redrawn as solid lines. Other examples of graphs and their medial graphs will be found throughout Chapters **??** and **??**.

If v is a 4-valent vertex of $\mathcal{M}$, an edge uv of $\mathcal{M}$ has a *direct extension* vw if the edges uv and vw separate the other two edges incident to the vertex v. A path $u_0u_1 \ldots u_k$ in $\mathcal{M}$ is called a *geodesic fragment* if each edge $u_{i-1}u_i$ has edge u_iu_{i+1} as a direct extension. A geodesic fragment $u_0u_1 \ldots u_k$ is called a *geodesic* if either of the following conditions holds.

(1) u_0 and u_k are points on the circle C.

(2) $u_k = u_0$ and $u_{k-1}u_k$ has u_0u_1 as direct extension.

Example 8.2 For the graph in Figure **??**a the four geodesics are:

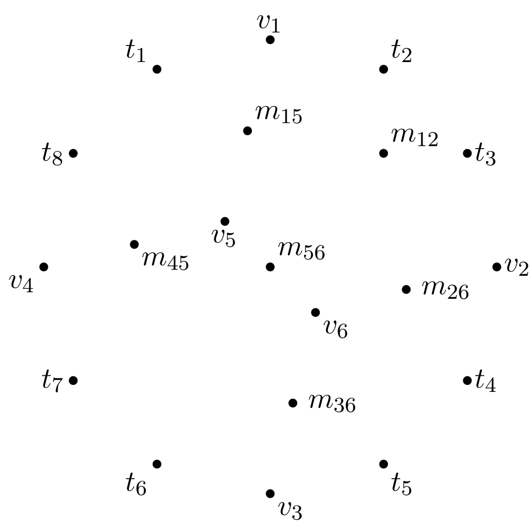
$$\begin{aligned}
\sigma_1 &= t_1m_{1,5}m_{5,6}m_{6,3}t_5 \\
\sigma_2 &= t_2m_{1,2}m_{2,6}m_{6,3}t_6 \\
\sigma_3 &= t_3m_{1,2}m_{1,5}m_{4,5}t_7 \\
\sigma_4 &= t_4m_{2,6}m_{6,5}m_{5,4}t_8
\end{aligned}$$

It is necessary to consider graphs that are similar to the graphs of geodesics that arise as medial graphs of a circular planar graph G, but are somewhat more general. Let D be the unit disc, with boundary circle C. An *arc* is a curve described by a differentiable function α, from $[0, 1]$ to the disc D, such that:

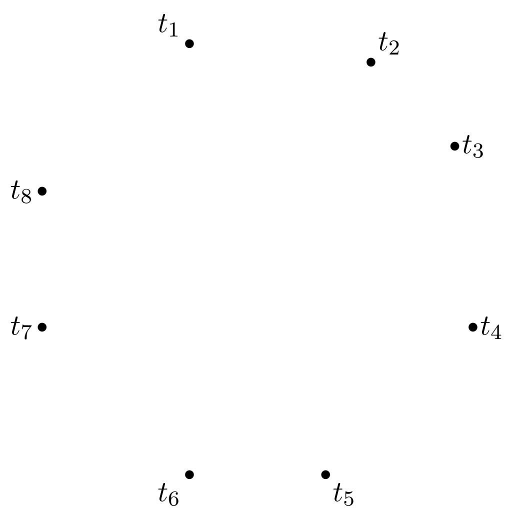
(1) for $0 < t < 1$, $\alpha(t)$ is in the interior of D.

(2a) the endpoints $\alpha(0)$ and $\alpha(1)$ are distinct points on C OR

(2b) $\alpha(0) = \alpha(1)$ and this point is in the interior of D.



(a)



(b)

Figure 8-1: Graph and Medial graph

(3) α has at most a finite number of self-intersection points, each of which is transversal.

A *family of arcs* in D is a finite set of arcs $\mathcal{A} = \{\alpha_i\}$ such that each intersection point lies on at most two arcs and each such intersection is transversal. If $\alpha(t_1)$ and $\alpha(t_2)$ are two intersection points (not necessarily adjacent) on an arc α, the set of points $\alpha(t)$ for $t_1 \leq t \leq t_2$, is called an *arc fragment*. If $\alpha(t_1)$ and $\alpha(t_2)$ are adjacent intersection points on α, the arc fragment between $\alpha(t_1)$ and $\alpha(t_2)$ is called an *arc segment*, or simply a segment.

Suppose G is a circular planar graph, and the medial graph $\mathcal{M} = \mathcal{M}(G)$ is constructed as above. After a small alteration in the geodesics in the neighborhood of each vertex of the form m_e the geodesics can be made into differentiable curves which intersect transversally, and $\mathcal{M}$ will be a family of arcs in D. For example, the geodesics in medial graph of Figure **??**a have been altered to become the family of arcs in Figure **??**b. Since they are topologically equivalent, we may consider either the family of geodesics formed by the medial lines, or the family of arcs to be the medial graph $\mathcal{M}(G)$ associated to G.

8.2 Coloring the Regions

Let $\mathcal{A} = \{\alpha_i\}$ be a family of arcs in D. A family $\mathcal{L}$ of closed loops can be made from $\mathcal{A}$ as follows. Suppose the endpoints of the arcs which meet the circle C are $t_1, t_2, \ldots t_{2n}$ numbered in circular order around C. For each $1 \leq i \leq n$, place a small curve (a semi-circle will do) from t_{2i-1} to t_{2i}. This gives a family of closed loops in the plane, all of whose intersections are 4-valent. By [**?**], Theorem 6.1.3, the regions of the plane may be 2-colored, say black and white. For definiteness, let the unbounded region be colored white. Figure **??** shows a 2-coloring of the regions defined by such a family of loops. Even if the arcs have no self-intersections, not every family of arcs $\mathcal{A}$ in D is the medial graph of a circular planar graph G, in any natural way, as shown both in Figure **??**.

If $\mathcal{A}$ is a family of arcs in the disc D, there is a cell decomposition K of D obtained by restricting the regions formed by the family of closed loops $\mathcal{L}$ to D. Specifically, the vertices of K are the points of intersection of one

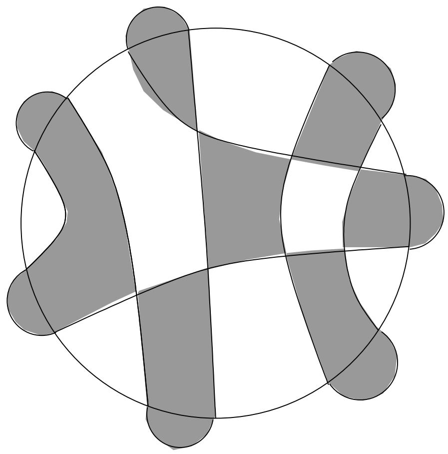


Figure 8-2: 2-coloring the regions

arc with another (or with itself), together with the points of intersection of an arc with the boundary circle C. An edge of K is the arc segment between adjacent vertices, or the segment on C between adjacent vertices. The complement of the edges of K in the disc D is a finite union of open connected regions. Each region is the connected component bounded by a finite number of edges. If R is a region in K, the intersection $\overline{R} \cap C$ is either empty or $\overline{R} \cap C$ consists of a finite number of edges of K. Each edge in $\overline{R} \cap C$ will be called a *boundary interval.* A region R is called an *interior region* if $\overline{R} \cap C$ is empty, that is, if R has no boundary intervals.

The 2-coloring of the regions inside the family $\mathcal{L}$ of closed loops constructed above, restricts to a 2-coloring of the regions of K. The boundary intervals may be called *black intervals* or *white intervals* as the case may be. The union of the black regions and black (boundary) intervals of K will be denoted by $\mathrm{bl}(K)$. The union of the white regions and white (boundary) intervals of K will be denoted by $\mathrm{wh}(K)$.

If $\mathcal{M}$ is the medial graph derived from a circular planar graph G, then G may be reconstructed from $\mathcal{M}$ as follows. The family $\mathcal{M}$ defines a cell decomposition K of the disc D which is 2-colored as above. This 2-coloring of K can be chosen so that the boundary nodes of G are in the black intervals of K. For each black region R_i, place a vertex p_i in its interior. For each

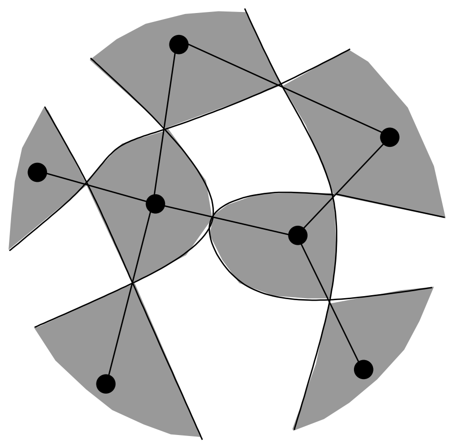


Figure 8-3: Reconstructed graph

pair of black regions R_i and R_j which have a common vertex w of K, place an edge $p_i p_j$ passing through w. The resulting graph is isomorphic to the original graph. In this case, the white complex wh(K) gives rise to the dual graph $G^{\perp}$ also considered as a circular planar graph embedded in D. The graph G_5 of Chapter **??** and its dual $G_5^{\perp}$ are shown in Figure **??**, where they are named $D(5,2)$ and $D(5,2)^{\perp}$ respectively.

Example 8.3 Figure **??** shows a circular planar graph, the medial graph, the coloring of the black regions and the reconstructed graph which is isomorphic to the original graph.

8.3 Switching Arcs

Let $\mathcal{A}$ be a family of arcs in the disc D and suppose $\{f, g, h\}$ are three arcs in $\mathcal{A}$ which intersect at u, v, and w to form a triangle $\triangle(uvw)$, such that there are no other vertices of $\mathcal{A}$ within a circle containing Δuvw, as shown in Figure **??**a. The region interior to $\Delta(u,v,w)$ is called an *empty triangle*.

f g f' g'

u v w'

h v' u' h'

w

(a) (b)

Figure 8-4: Switching arcs

f g f g

p q p q

h h

r r

(a) (b)

Figure 8-5: $Y - \triangle$ Transformation; Switching arcs

A *switch* of $\{f, g, h\}$ consists of replacing this configuration with that of Figure **??**b where the arcs have been renamed $\{f', g', h'\}$ respectively. After the switch, the region in Figure **??**b interior to $\Delta(u', v', w')$ is also an empty triangle. Performing the switch again to the arcs of Figure **??**b, the triple $\{f', g', h'\}$ will revert to the triple $\{f, g, h\}$ Figure **??**a. $\mathcal{A}$ and $\mathcal{A}'$ are said to be *equivalent* families of arcs if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by a finite sequence of switches. Each $Y - \triangle$ transformation of G corresponds to a switch of a triple of arcs in $\mathcal{M}$. Conversely, a switch of a triple of arcs in $\mathcal{M}$ corresponds to a $Y - \triangle$ transformation of G as illustrated in Figure **??**. The result is stated in the following theorem.

Theorem 8.1 *Two circular planar graphs are $Y - \triangle$ equivalent if and only if their medial graphs are equivalent as families of arcs.*

8.4 Lenses

Suppose $\mathcal{A}$ is a family of arcs in the disc D, which intersect at (distinct) points p and q. A subgraph L of the graph formed by the family $\mathcal{A}$ is called a *lens* if the following two conditions are satisfied.

(i) There are two arc fragments $a_0a_1 \ldots a_lb_0$ and $b_0b_1 \ldots b_ma_0$. The sequence of arc segments

$$\mathcal{P} = a_0a_1 \ldots a_lb_0b_1 \ldots b_ma_0$$

is a simple closed path in the interior of D.

(ii) L consists of the vertices and arc segments of $\mathcal{P}$ together with all vertices and arc segments of $\mathcal{A}$ in the interior of the bounded component of the complement of $\mathcal{P}$.

Observation 8.1 (1) The only repeated vertices are a_0 and b_0, called the *poles* of the lens. It can happen that $b_0 = a_0$ (that is, there are no points b_i for $i > 0$). In this case, there is only one pole a_0, and the path $\mathcal{P}$ is $a_0a_1 \ldots a_la_0$ and $\mathcal{P}$ will be called a *loop* in $\mathcal{A}$. This degenerate case of a lens is handled in much the same way as a true lens which has two poles.

Example 8.4 Figure **??** illustrates an example of a lens L. The arcs g and h intersect at p and q which are the poles of L. The closed path is $\mathcal{P} = pv_3v_2v_1qu_3u_2u_1p$. Arcs g and α_1 do not form a lens, nor do h and α_1 form a lens. There are fragments of arcs α_1, α_2 and α_3 within the lens L.

This proof of the following lemma is an adaptation to our situation, of a proof due to Grunbaum. Not only is the statement itself important, but the method of proof will be used extensively in later arguments. Grunbaum constructed medial graphs and used them to prove a theorem of Steinitz, called "The Fundamental Theorem of Convex Types". In [**?**], the definition of lens is slightly more restrictive than that given here, because in the proof of Steinitz's Theorem, it is necessary to find a lens for which no edges in the interior of the lens are incident to the poles. See [**?**] and [**?**].

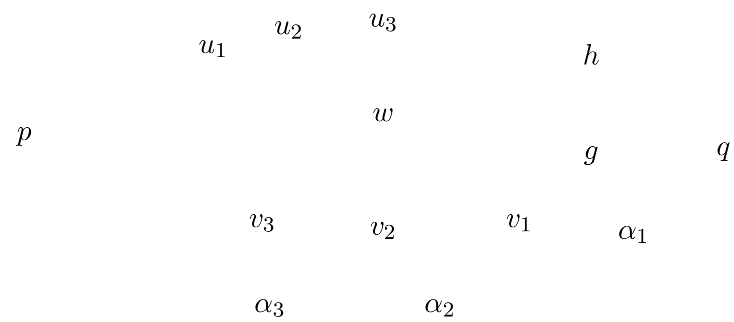


Figure 8-6: Lens with arcs crossing

Lemma 8.2 *Suppose $\mathcal{A}$ is a family of arcs which has a lens. Then $\mathcal{A}$ is equivalent to a family of arcs which has an empty lens.*

Proof: Suppose that L is a lens for which the number of regions inside L is minimal. Suppose p and q are the two poles of L and suppose $g = pp_1 \dots p_l q$ and $h = qq_1 \dots q_m p$ are two arc fragments which form the closed path $\mathcal{P}$ which is the boundary of L. (The situation is similar if L has only one pole p. The details of this case are left to the reader.) Any arc that crosses g or h between p and q must cross both g and h between p and q or there would be a lens with fewer regions than L. Let $\mathcal{F} = \{\alpha_i\}$ be the set of arcs in $\mathcal{A}$ which intersect both g and h between p and q. For each $i = 1, \dots, m$, let v_i be the point of intersection of α_i with g between p and q. Let w_i be first intersection point of α_i with another member of $\mathcal{F}$ after v_i inside the lens L. Let $W = \{w_i\}$ be the set of points obtained in this way. If W is empty, let α_i be the arc in $\mathcal{F}$ such that v_i is closest to p. Then arcs $\{g, h, \alpha_i\}$ form an empty triangle with p as one vertex. A switch of $\{g, h, \alpha_i\}$ will remove α_i from the lens L. If W is nonempty, each point $w_i \in W$ is the point of intersection of two arcs in $\mathcal{F}$. Let w be a point in W for which the number of regions within the configuration formed by α_i, α_j and g is a minimum. This minimum must be one, or else there would be another arc which intersects α_i between v_i and w or which intersects α_j between v_j and w. Then $\{g, \alpha_i, \alpha_j\}$

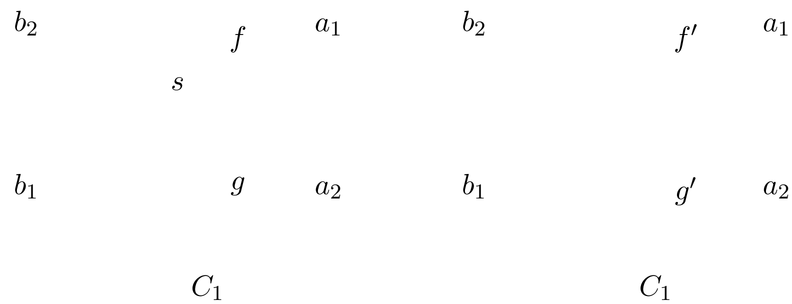
form an empty triangle. A switch of $\{g, \alpha_i, \alpha_j\}$ will reduce the number of regions within the lens L. After a finite number of switches, there are no intersections of any members of $\mathcal{F}$ with each other within L. After finitely many more switches, no member of $\mathcal{F}$ crosses into L, and the newly placed family has an empty lens.

Example 8.5 Refer to Figure **??**. The arc fragments g and h are the boundary path $\mathcal{P}$ of a lens with six interior regions. Each of the arcs α_1, α_2 and α_3 intersects both g and h. Arcs α_1 and α_2 intersect at w. Arcs α_1, α_2 and g form an empty triangle $\Delta w_1 v_1 v_2$. After a switch of $\{\alpha_1, \alpha_2, g\}$, the arc α_2 forms an empty triangle with g and h. A switch of $\{\alpha_2, g, h\}$ then produces in a lens with only two arcs α_1 and α_3, each of which intersects both g and h. The (newly placed) arcs α_1 and α_3 form another empty triangle with g. After a switch of $\{\alpha_1, \alpha_3, g\}$, the arcs α_1 and α_3 no longer intersect within the lens. A switch of $\{\alpha_1, g, h\}$ and then a final switch of $\{\alpha_3, g, h\}$ will empty the lens.

8.5 Uncrossing Arcs

Suppose $\mathcal{A}$ is a family of arcs in D, and f and g are two arcs in $\mathcal{A}$ which intersect at s. An *uncrossing* of f and g at s, will result in a new family of arcs $\mathcal{A}'$ as follows. The only arcs affected are f and g, which will be replaced by arcs f' and g'. First make a small circle C_1 around s which contains no portions of arcs other than f and g, and which contains no intersection points of f and g other than s. Suppose $f = x_1 \dots a_1 s b_1 \dots y_1$ where x_1, y_1 are the endpoints of f; a_1, b_1 are the points where f crosses C_1. Similarly, $g = x_2 \dots a_2 s b_2 \dots y_2$ where x_2, y_2 are the endpoints of g; a_2, b_2 are the points where g crosses C_1. Two crossed arcs f and g are shown in Figure **??**a. To uncross f and g, first eliminate the vertex s, and eliminate the two arc fragments $a_1 s b_1$ and $a_2 s b_2$. Next, place an arc segment from a_1 to b_2 and an arc segment from a_2 to b_1 each inside the circle C_1 and in such a way that these two arc segments don't cross. The new arc f' is the path $x_1 \dots a_1 b_2 \dots y_2$. The new arc g' is the path from $x_2 \dots a_2 b_1 \dots y_1$. The uncrossed arcs are illustrated in Figure **??**b.

There are two possible uncrossings at each intersection point. A choice can be made by specifying how the points a_1, b_1, a_2, b_2 are to be re-joined



(a) Arcs crossed (b) Arcs uncrossed

Figure 8-7: Uncrossing two arcs

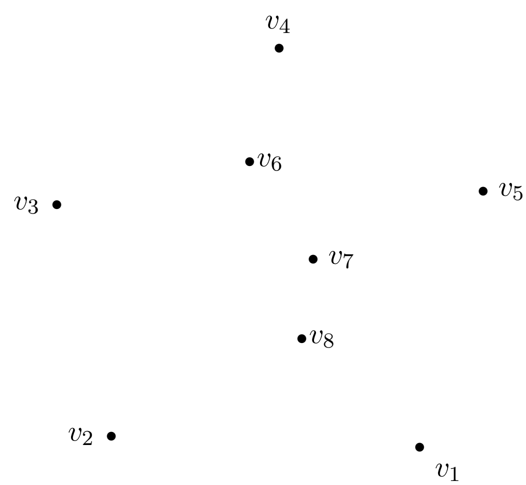
within C_1. If f and g intersect only once, the choice can be made by specifying the endpoints of the new arcs f' and g' in terms of the original endpoints of f and g. If f and g intersect more than once in $\mathcal{A}$ to form a lens (or more than one lens), f' and g' will still intersect in $\mathcal{A}'$.

Figure **??**a shows a graph G, and Figure **??**b its medial graph $\mathcal{M}$. The geodesics in the medial graph will be named by (the indices of) their endpoints. The geodesics in the medial graph $\mathcal{M}$ in Figure **??**b are $(1,7)$, $(2,6)$, $(3,8)$, $(4,10)$ and $(5,9)$. The intersection of arcs $(3,8)$ and $(4,10)$ is the point s, which corresponds to the edge v_7v_8 in the original graph G.

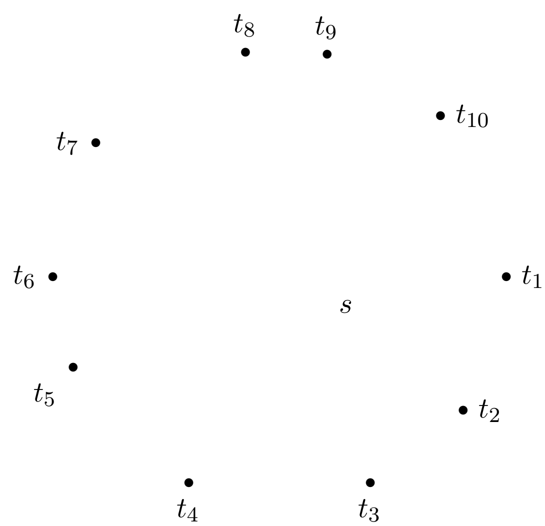
(1) Suppose edge v_7v_8 is contracted to a single node w and the result is a new graph G', as shown in Figure **??**a. This contraction of v_7v_8 in G corresponds to an uncrossing of $(3,8)$ and $(4,10)$ producing geodesics $f' = (3,10)$ and $g' = (4,8)$ in the medial graph $\mathcal{M}'$. The medial graph $\mathcal{M}'$ is shown in **??**b.

(2) Suppose the edge v_7v_8 is deleted from the graph G. This corresponds to the other uncrossing of geodesics $f = (3,8)$ and $g = (4,10)$ in $\mathcal{M}$, producing new geodesics $f'' = (3,4)$ and $g'' = (8,10)$ in $\mathcal{M}''$. The geodesics of $\mathcal{M}''$ are $(1,7), (2,6), (3,4), (5,9), (8,10)$. The figures illustrating this case are left to the reader.

Example 8.6 Figure **??**a shows a graph G with 4 boundary nodes, 3 inte-

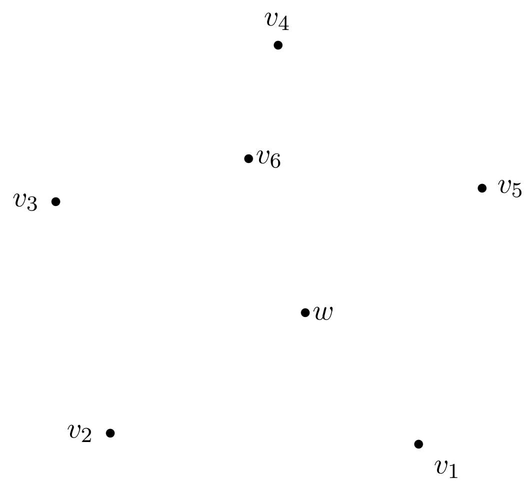


(a) Graph

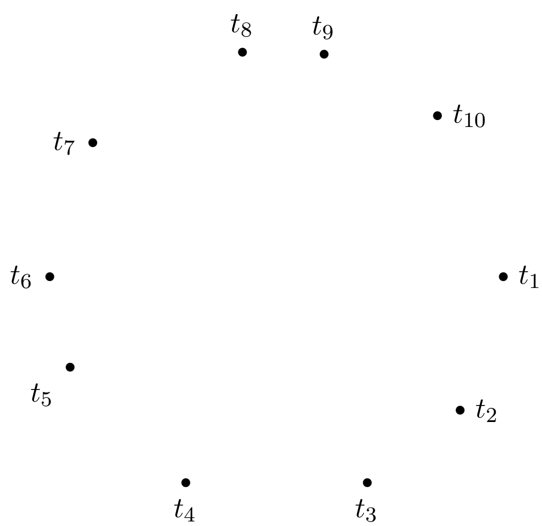


(b) Medial Graph

Figure 8-8: Graph and Medial graph



(a) Edge contracted



(b) Geodesics uncrossed

Figure 8-9: Edge contracted and geodesics uncrossed rior nodes, and 7 edges; the edges are indicated by solid lines. The geodesics of the medial graph $\mathcal{M}$, are indicated by dotted lines. Figure **??**b shows the medial graph, now indicated by solid lines, without the original graph G. The points $t_1, \ldots, t_8$ have been placed on the boundary circle according to Section **??**. The geodesics are named by the indices of their endpoints. The corner in the middle of the geodesic $(3, 6)$ has not been smoothed. The geodesics $(2, 7)$ and $(3, 6)$ in the medial graph $\mathcal{M}$ intersect twice at s and t to form a lens. In graph G, the corresponding edges v_5v_6 and v_6v_7 are in series, as in Chapter **??**, Figure **??**. Suppose a modification is made as in Figure **??**, replacing these two edges by a single edge v_5v_7. The result is the graph G' in Figure **??**a. In the medial graph $\mathcal{M}$, this corresponds to an uncrossing of arcs $(2, 7)$ and $(3, 6)$ at one of the intersection points, and the result is the medial graph $\mathcal{M}'$, shown in Figure **??**b. The medial graph $\mathcal{M}'$ no longer has a lens. The corners of the geodesics $(2, 6)$ and $(3, 7)$ $\mathcal{M}'$ have not been smoothed.

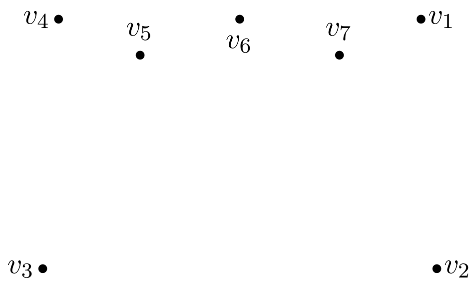
8.6 Families of Chords

An arc which begins and ends on the boundary circle C and has no self-intersection, is called a *chord*. If α is a chord, the points $\alpha(0)$ and $\alpha(1)$ are called the *endpoints* of α. Since the endpoints uniquely identify the chord, we can (and will) use $(\alpha(0), \alpha(1))$ as the name for the chord α. The crossings of one chord with another are not indicated by their endpoints, so the set of pairs $(\alpha_i(0), \alpha_i(1))$ do not uniquely specify the family $\mathcal{A} = \{\alpha_i\}$.

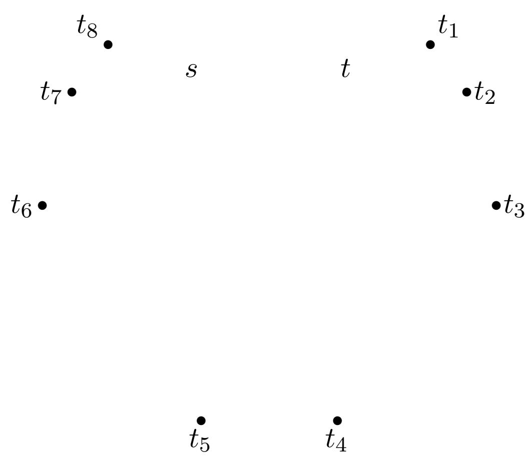
If $\mathcal{A}$ is a family of chords in D, and $\mathcal{A}$ does not have a lens, nor a degenerate lens formed by a loop, we say that $\mathcal{A}$ is *lensless*.

Theorem 8.3 *Suppose that $\mathcal{A}$ is a family of arcs that has one or more lenses. Then by a finite sequence of switches and uncrossings of arcs that form lenses, $\mathcal{A}$ can be reduced to a family that is lensless.*

Proof: If $\mathcal{A}$ has a lens, let L be a lens for which the number of regions inside L is minimal. Lemma **??** shows that a finite number of switches will make L empty. Each switch preserves the number of intersections of arcs. An uncrossing at a pole reduces the number of intersections of arcs by one. After a finite number of emptying lenses and uncrossings at a pole, the process must stop, yielding a family that is lensless.

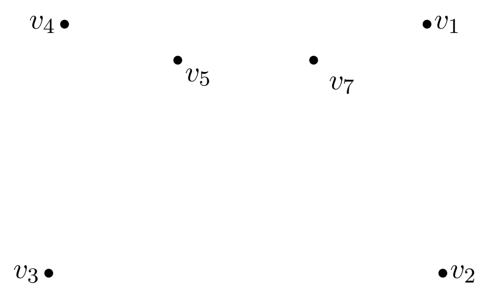


(a) Graph with edges in series

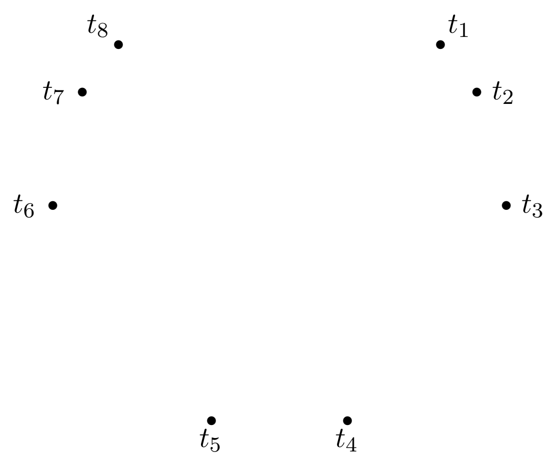


(b) Medial graph with lens

Figure 8-10:



(a) Edge contracted



(b) Lens removed

Figure 8-11: Edge in G contracted; Lens removed from $\mathcal{M}$

Suppose $\mathcal{A}$ is a family of arcs and $\{h, g\}$ is a pair of arcs which intersect at w. Suppose u is an endpoint of h and v is an endpoint of g, and that there are no other vertices of $\mathcal{A}$ in the closed region bounded by the segments $\widehat{uv}$ $\widehat{vw}$ $\widehat{wu}$ as in Figure **??**a. Such a configuration is called an *empty boundary triangle.* $\Delta(vxy)$.

Lemma 8.4 *Suppose $\mathcal{A}$ is a family of chords, and that h and g are two chords $\mathcal{A}$ which intersect at w in the interior of the disc D. Let x be an endpoint of h and let y be an endpoint of g. Suppose that $\mathcal{F} = \{f_1, \ldots, f_m\}$ is the set of chords in $\mathcal{A}$ intersects h between w and x. Suppose that for each $1 \leq i \leq m$, f_i also intersects g between w and y. Then a finite sequence of switches will remove all members of $\mathcal{F}$ from the sector xwy.*

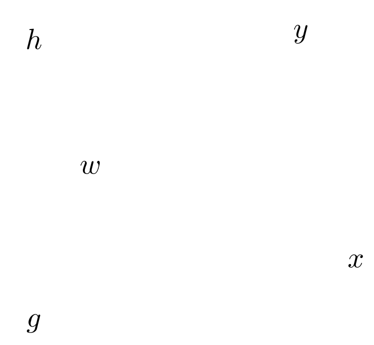
Proof: Instead of giving the proof of Lemma **??**, which closely follows the proof of **??**, and the proof in [**?**], p. 239, the method will be illustrated by an example.

Example 8.7 In Figure 9-9b, arcs g and h intersect at w to form a triangle Δwxy which is not empty. Arcs f_1 and f_2 intersect at z within triangle wxy. Arcs h, f_1, and f_2 form an empty triangle. A switch of arcs $\{h, f_1, f_2\}$ reduces the number of intersections inside triangle wxy by one. Then a switch of $\{g, h, f_2\}$ reduces the number of arcs that enter the sector. Finally, a switch of $\{g, h, f_1\}$ makes Δwxy into an empty triangle.

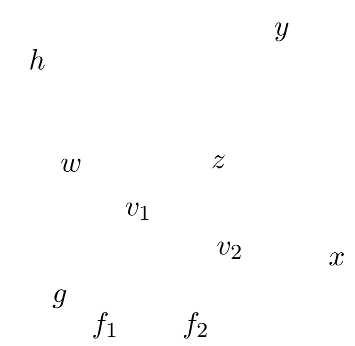
Lemma 8.5 *Suppose $\mathcal{A}$ is a family of chords. Suppose f, g and h are three chords, such that f and g intersect at p, f and h intersect at q, g and h intersect at r. Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be the set of chords in $\mathcal{A}$ each of which intersects chord h between r and q. Suppose that for each $1 \leq i \leq m$, f_i also intersects chord g between p and r. Then a finite sequence of switches will remove all members of $\mathcal{F}$ from the sector qpr. The switches can be chosen so that chord f is never involved.*

Proof: The proof of Lemma **??** is exactly the same as for Lemma **??**, with the boundary circle C replaced by chord f.

Suppose $\mathcal{A}$ is a family of chords in D. Suppose f, with endpoints x_1 and y_1; and g, with endpoints x_2 and y_2, are two chords in $\mathcal{A}$ which intersect at



(a) Empty Triangle



(b) Triangle, with arcs

Figure 8-12: Emptying a boundary triangle

v to form with C an empty boundary triangle $\Delta x_1 v y_1$, An *uncrossing* of $\mathcal{A}$ at v will result in a new family of chords $\mathcal{A}'$ where f' has endpoints x_1 and y_2; g' has endpoints x_2 and y_1. This will give a new cell complex K', with one less region than K. If a coloring of the cell complex K is chosen, then K' is to be colored consistently with the coloring of K.

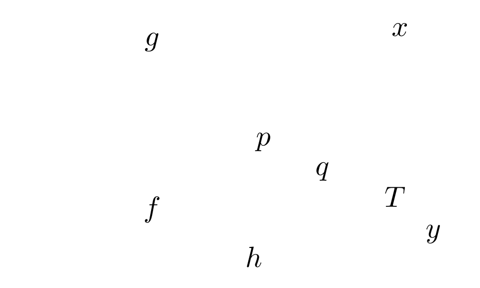
The new arc f' with endpoints x_1 and y_2 has no self-intersections, because the chords f and g in $\mathcal{A}$ did not intersect in segment vx_1. Every other chord h is unchanged by the uncrossing, and has no intersection with f between v and x. Therefore f' cannot form a lens with h. Similarly, g' has no self-intersections and cannot form a lens with any other chord h. Finally, f' and g' do not form a lens, because they do not intersect at all. The result of this uncrossing at the vertex v will result in a new family of chords $\mathcal{A}'$ which is also lensless.

Lemma 8.6 *Suppose that $\mathcal{A}$ a lensless family of chords in D, and assume that each chord in $\mathcal{A}$ intersects at least one other chord in $\mathcal{A}$. Then there are at least three empty boundary triangles with disjoint interiors.*

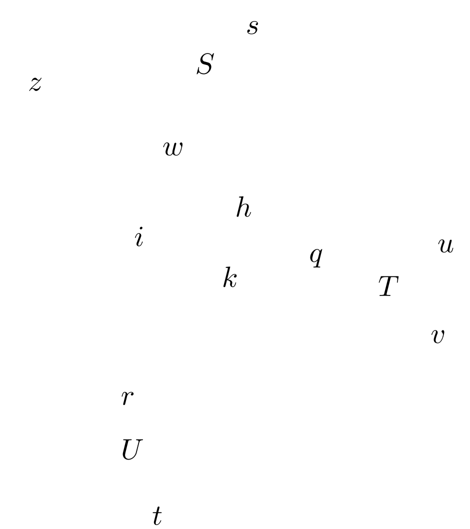
Proof: This is another (slightly different) use of Grunbaum's method of proof. Suppose f and g are two chords which intersect at p. We may assume that x is an endpoint of f, and y is an endpoint of g and there is no vertex of f between p and x, as in Figure **??**a.

If there is a vertex on arc fragment py, take q to be the vertex on g which is the closest to y, and let h be the chord in $\mathcal{A}$ which crosses g at q. One endpoint of h must be on the segment $\widehat{xy}$, by the choice of p, and because $\mathcal{A}$ is lensless. Continuing in this way, after a finite number of steps, we find two chords say h and k which intersect at q to form a boundary triangle Δquv within Δpxy, and such that there are no vertices on qu or qv. If Δquv is empty, it is the first of the required empty boundary triangles T. Otherwise, there must be at least two chords which are entirely within Δquv, and which intersect each other. Proceeding as above a triangle is found within Δquv, and eventually the first empty boundary triangle T, which is inside Δpxy. In Figure **??**b, the chords h and k intersect at a vertex q so that $T = \Delta quv$ is the first empty boundary triangle.

One endpoint of h is v; suppose the other endpoint is z. If there are no vertices of $\mathcal{A}$ on arc fragment qz, $S = \Delta qzu$ is the second triangle. Otherwise, suppose that i is the chord which crosses qz at w, with w the



(a) First boundary triangle



(b) Three boundary triangles

Figure 8-13: Locating three boundary triangles vertex of $\mathcal{A}$ on qz closest to z. Suppose the endpoints of i are s and t, with s on the arc uz of the boundary circle C. By an argument similar to that above, a second empty boundary triangle S is found inside Δzsw.

Finally, consider Δwzt. Let r be the vertex on wt closest to t. By another argument similar to the one above, a third empty boundary triangle U is found within Δrzt. This is illustrated in Figure 9-12b.

8.7 Standard Arrangements

Let $\{t_1, t_2, \ldots, t_{2n}\}$ be distinct points in circular order on the circle C. For notational convenience the point t_i will be identified with the number i, and the set of points $\{t_1, t_2, \ldots, t_{2n}\}$ will be identified with the set of numbers $\{1, 2, \ldots, 2n\}$. Suppose $\{1, 2, \ldots, 2n\}$ is partitioned into a set S consisting of n pairs of integers. That is, $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ is a set of n pairs of integers where each integer from 1 to $2n$ occurs exactly once. Assume the indexing is arranged so that for each $1 \leq i \leq n$, $x_i < y_i$, and if $x_i < x_j$ then $i < j$. A family $\mathcal{S} = \{\sigma_i\}$ of chords will be placed in the disc D so that for each $1 \leq i \leq n$, the endpoints of σ_i are x_i and y_i, and the points of intersection on σ_i occur in a certain order between the points x_i and y_i. A consequence of Theorem **??** (to come) is that any other family of chords with the same endpoints as $\mathcal{S}$ is equivalent (by switches) to $\mathcal{S}$.

Given the set $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, the pairs (x_i, y_i) and (x_j, y_j) are said to be *interlaced* if the numbers x_i, y_i, x_j, y_j occur in the order $x_i < x_j < y_i < y_j$. If (x_i, y_i) and (x_j, y_j) are interlaced, the chords σ_i and σ_j must intersect at some point, say $p(i, j)$, in D. The chords will be placed in D so that the intersection points $p(i, j)$ on each σ_i occur in a certain order. Specifically, the order is that if $p(i, j)$ and $p(i, k)$ are two intersection points on σ_i with $i < j < k$, then the point $p(i, j)$ is closer to x_i than $p(i, k)$ is to x_i. The result of placing the chords in this way will be a lensless family $\mathcal{S} = \{\sigma_i\}$, called the *standard arrangement* of the chords σ_i determined by the set $\mathcal{S}$. The placing of the chords is as follows.

(1) A chord σ_1 is placed joining x_1 to y_1.

(2) Assume inductively for each $1 \leq i < m \leq n$, that the chord σ_i has been placed in D. Then the chord σ_m will be placed in D as follows. If (x_m, y_m) interlaces the pair (x_i, y_i), with $i < m$, then a point $p(i, m)$ is to

be put on σ_i closer to y_i than any of the points $p(i,j)$ previously placed on σ_i. If (x_m, y_m) does not interlace (x_i, y_i), there is to be no point $p(i,m)$. Now join x_m to y_m by a chord, which for each $1 \le i < m$, intersects σ_i transversally at $p(i,m)$ and at no other point of σ_i. The order in which the points $p(i,j)$ occur on σ_i will be called the *standard order.*

Theorem 8.7 *Suppose that $\mathcal{A} = \{\alpha_i\}$ and $\mathcal{B} = \{\beta_i\}$ are two lensless families of n chords in the disc D, and for each i, the endpoints of α_i are the same as the endpoints of β_i. Then $\mathcal{A}$ and $\mathcal{B}$ are equivalent by switches of chords.*

Proof: Let $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, be the set of pairs of endpoints, which is assumed to be the same for both $\mathcal{A}$ and $\mathcal{B}$. Assume that the pairs in the set S are ordered as above. $\mathcal{A}$ will be shown to be equivalent to the standard family $\mathcal{S}$. For each i and j with $i < j$, if (x_i, y_i) interlaces (x_j, y_j), let $a(i,j)$ denote the point of intersection of chord α_i and chord α_j in $\mathcal{A}$.

Start with chord α_1. Suppose some intersection point precedes $a(1,2)$ on α_1; assume that $a(1,k)$ is the immediate predecessor of $a(1,2)$ on α_1. Then α_k must intersect α_2 between x_2 and $a(1,2)$. Consider the three chords α_2, α_k and α_1 in the role of f, g and h of Lemma **??**. A finite sequence of switches, not involving α_1, will remove all crossings into the triangle formed by α_2, α_k and α_1. Then a switch of $\{\alpha_2, \alpha_k, \alpha_1\}$ will have the result that $a(1,2)$ now precedes $a(1,k)$ on α_1.

Continuing in this way, make switches so that all of the intersection points $a(1,j)$ which occur are in standard order on α_1. Next, without involving α_1, make switches so that all points $a(2,j)$ in standard order on α_2. Continue with α_3, α_4, etc. until all crossings on all chords are in standard order. The family $\mathcal{A}$ has been shown to be equivalent to the family $\mathcal{S}$ in standard position. Similarly $\mathcal{B}$ is equivalent to $\mathcal{S}$ so $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Example 8.8 Figure **??** shows two examples of families of chords in the disc, each with 5 chords. The chords will be named by their endpoints. The 5 pairs of endpoints, (the same for each family) is

$$S = \{(1,6), (2,8), (3,7), (4,10), (5,9)\}$$

The family $\mathcal{A}$ in Figure **??**a is not in standard position, because the intersection point of chord $(4, 10)$ and $(1, 6)$ is closer to t_1 than the intersection of $(3, 7)$ and $(1, 6)$. The family $\mathcal{B}$ *is* in standard position. Let $f = (1, 6)$, $g = (3, 7)$ and $h = (4, 10)$. Chords $\{f, g, h\}$ form an empty triangle, and a switch of $\{f, g, h\}$ will change $\mathcal{A}$ to $\mathcal{B}$.

Let

$$S_n = \{(1, n+1), (2, n+2), \ldots, (n, 2n)\} \tag{8.1}$$

and let $\mathcal{S}_n$ be the family of arcs in standard position with endpoint set S_n. A realization of the family $\mathcal{S}_7$ is shown in Figure **??**, where the points $t_1, t_2, \ldots, t_{14}$ are in clockwise order on the boundary "circle" which is indicated by the dashed lines. The complex K is formed from $\mathcal{S}_n$ as in Section **??**. The 2-coloring of the regions of K is chosen so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < t_4 < \ldots < t_{2n-1} < v_n < t_{2n} < t_1$$

where $v_1, \ldots, v_n$ are in the black intervals. The complex $\mathrm{bl}(K)$ gives rise to a graph H_n as in Section **??**. A realization of the graph H_7 is shown in Figure **??**.

For $n = 2m + 1$ an odd integer, the graph H_n is described as follows.

(1) The nodes of H_n are the integer lattice points (i, j) which satisfy

For each $1 \le j \le m + 1$, $j \le i \le 2m + 2 - j$.

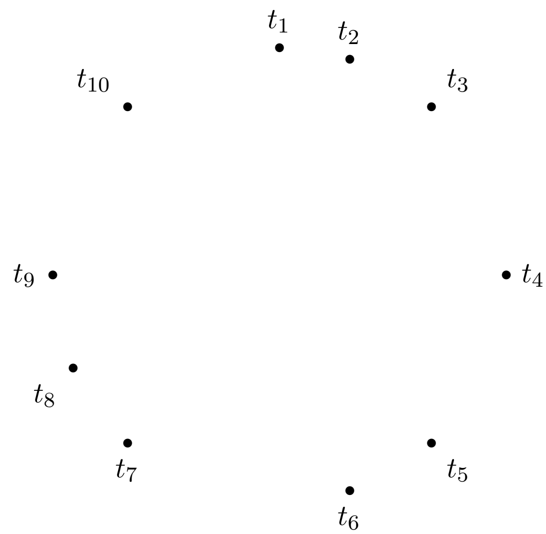
(2) The edges of H_n are the horizontal or vertical line segments of length 1 joining adjacent nodes.

(3) The boundary nodes of H_n are the nodes of the form $v_i = (i, i)$ for $1 = 1, 2, \ldots, m + 1$ and the nodes of the form $v_i = (i, 2m + 2 - i)$ for $i = m + 2, \ldots, 2m + 1$.

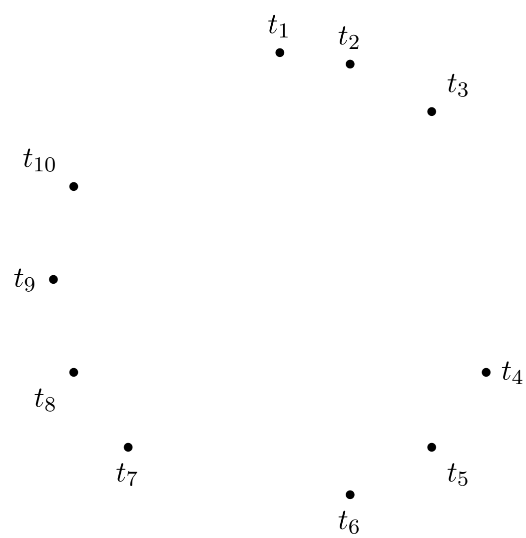
There is a similar graph H_n for each even integer. The details are left to the reader.

Lemma 8.8 *The graph H_n is (1) critical and (2) well-connected.*

Proof: (1) The medial graph for H_n is the family of arcs S_n constructed above, which is lens-free. Anticipating Lemma **??**, which will be proven in Section **??**, this implies that H_n is critical.



(a) Family of chords



(b) Chords in standard position

Figure 8-14: Switching chords into standard position

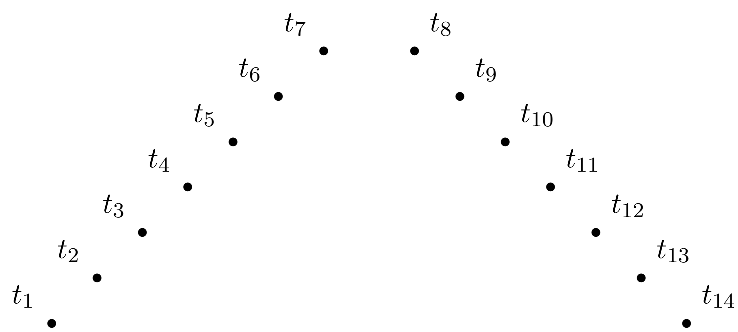


Figure 8-15: Standard family $\mathcal{S}_7$

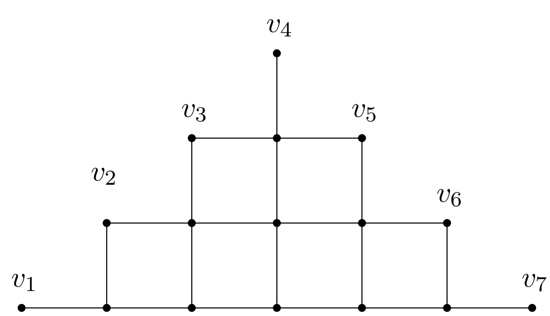


Figure 8-16: Standard graph H_7

(2) Suppose $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is a circular pair of vertices. By reversing the roles of P and Q and re-indexing, if necessary, assume that $p_1 = v_1$, and p_k occurs before v_m in the clockwise circular order starting from v_1. Suppose the points in P are $p_j = (a_j, a_j)$, and the points in Q are $q_j = (b_j, c_j)$. Then p_j will be joined to q_j by the path α_j as follows.

(1) If $a_j \leq c_j$, $\alpha_j = (a_j, a_j) \to (b_j, a_j) \to (b_j, c_j)$

(2) If $a_j > c_j$, $\alpha_j = (a_j, a_j) \to (a_j, c_j) \to (b_j, c_j)$

It is easily seen that if $i \neq j$, then the paths α_i and α_j do not intersect, so the set $\alpha = \{\alpha_j\}$ is a connection $P \leftrightarrow Q$.

Corollary 8.9 *The graph G_n of Chapter* **??** *is both critical and well-connected.*

Proof: The geodesics for G_n do not form a lens, so G_n is critical by Lemma **??**. The endpoint set for the geodesics of G_n is the set S_n defined above in Equation **??**. Therefore G_n is $Y - \triangle$ equivalent to H_n, which is well-connected.

Observation 8.2 Suppose that $n = 2m$ and $(P;Q)$ is a circular pair of boundary nodes of H_n, where $P = (p_1, \ldots, p_m)$ and $Q = (q_1, \ldots, q_m)$. P (and Q) must consist of a set of m consecutive boundary nodes of H_n. Assume that $P = (v_1, v_2, \ldots, v_m)$. By inspection, the family of paths $\alpha = \{\alpha_i\}$ described in the proof of Lemma **??** is the only connection $P \leftrightarrow Q$. The connections in G_n are in $1 - 1$ correspondence with the connections in H_n. Thus if $(P;Q) = (p_1, \ldots, p_m; q_1, \ldots, q_m)$ is a circular pair in G_n, there is exactly one connection $P \leftrightarrow Q$. Since a principal flow path gives a connection, there can be only one principal flow path with the boundary data as in Section **??**.

In the case $n = 2m+1$ and $(P;Q) = (p_1, \ldots, p_m; q_1, \ldots, q_m)$ is a circular pair, then either P or Q must consist of a set of m consecutive boundary nodes of H_n. By reversing the roles of P and Q, and re-indexing if necessary, we may assume that $P = (v_1, v_2, \ldots, v_m)$. Then, similar to the case of n even, there is only one family $\alpha = \{\alpha_j\}$ of paths joining P to Q, and there is only one principal flow path with the boundary data as in Section **??**.

Chapter 9

Recovering a Graph

9.1 Connections

Let $\mathcal{A}$ be a family of arcs in the disc D, and let K be the cell complex constructed from $\mathcal{A}$ as in Section **??**. The regions of K are 2-colored, say black and white. Suppose e and f are two black intervals on the boundary circle C. The definition of a *path* from e to f through $\mathrm{bl}(K)$ is similar to the definition in Chapter **??** of a path from one boundary vertex to another through a graph G. The notation $(e \leftrightarrow f)$ of Chapter **??** will indicate the existence of a path from e to f.

- A *path* $(e \leftrightarrow f)$ from e to f in $\mathrm{bl}(K)$ consists of a finite sequence:

$$(e \leftrightarrow f) = e, B_1, B_2, \ldots, B_k, f$$

where

(1) The B_i are regions of $\mathrm{bl}(K)$;

(2) There is at least one B_i; no B_i occurs more than once;

(3) For each $1 \leq i < k$, B_i and B_{i+1} have a common vertex;

(4) e is a boundary interval of B_1; f is a boundary interval of B_k;

(5) Each region B_i other than B_1 and B_k is an interior region of $\mathrm{bl}(K)$.

Figure **??** shows a family of 7 chords which form a cell complex K in which there are 8 regions $R_1, \ldots, R_8$. The regions $R_1, \ldots, R_6$ are boundary regions; R_7 and R_8 are interior regions. There are 7 black intervals $e_1, \ldots, e_7$.

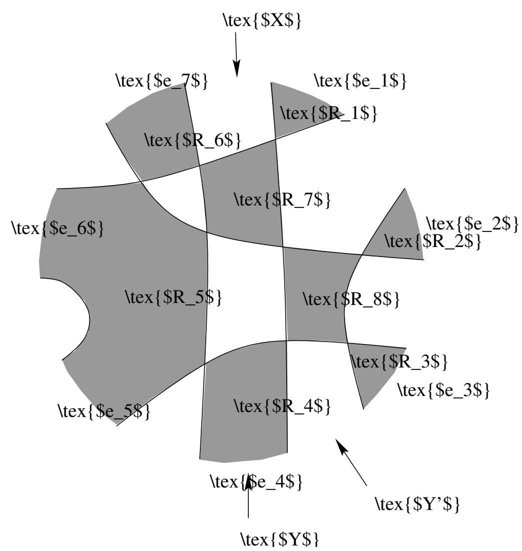


Figure 9-1:

In Figure **??**, there are the following paths between black intervals (and many others):

$$
\begin{aligned}
(e_1 \leftrightarrow e_7) &= e_1, R_1, R_7, R_6, e_7 \\
(e_2 \leftrightarrow e_3) &= e_2, R_2, R_8, R_3, e_3 \\
(e_4 \leftrightarrow e_5) &= e_4, R_4, R_5, e_5 \\
(e_5 \leftrightarrow e_6) &= e_5, R_5, e_6
\end{aligned}
$$

It follows from the definition that

(1) If there is a path $(e \leftrightarrow f)$, there is a simple continuous curve $\beta(t)$, where $\beta(0) \in e$, $\beta(1) \in f$, and for all $0 \leq t \leq 1, \quad \beta(t)$ is in the closure of the black regions of K.

(2) Let G be a circular planar graph and $\mathcal{M}$ be its medial graph. Suppose v_i and v_j are boundary nodes of G, and that e_i and e_j are the black intervals that correspond to them respectively. If there is a path between v_i and v_j through G, there will be a path between e_i and e_j through bl$K$, and conversely.

Suppose that there are two sets of black intervals $E = \{e_i\}$ and $F = \{f_i\}$, which are in circular order around C:

$$e_1 < e_2 < \ldots < e_k < f_k < f_{k-1} < \ldots < f_1$$

The definition of a *k-connection* through bl$K$ is similar to the definition in Chapter **??** of a k-connection through a graph G. Specifically,

- A *k-connection* $(E \leftrightarrow F)$ from E to F is a set of paths $\{(e_i \leftrightarrow f_i)\}$, for $1 \leq i \leq k\}$, such that all the (necessarily black) regions which occur are distinct.

A typical k-connection is displayed by the following scheme.

$$\begin{array}{rcl} (e_1 \leftrightarrow f_1) & = & e_1, B_{1,1}, B_{1,2}, \ldots B_{1,n_1}, f_1 \\ (e_2 \leftrightarrow f_2) & = & e_2, B_{2,1}, B_{2,2}, \ldots B_{2,n_2}, f_2 \\ \cdots & & \cdots \\ \cdots & & \cdots \\ (e_k \leftrightarrow f_k) & = & e_k, B_{k,1}, B_{k,2}, \ldots B_{k,n_k}, f_k \end{array}$$

For example in the preceding figure, the pair (e_1, e_2) is 2-connected to the pair (e_7, e_4), by the two paths:

$$\begin{array}{rcl} (e_1 \leftrightarrow e_7) & = & e_1, R_1, R_7, R_6, e_7 \\ (e_2 \leftrightarrow e_4) & = & e_2, R_2, R_8, R_4, e_4 \end{array}$$

Also (e_3, e_4) is 2-connected to (e_7, e_6), by the paths:

$$\begin{array}{rcl} (e_3 \leftrightarrow e_7) & = & e_3, R_8, R_7, R_6, e_7 \\ (e_4 \leftrightarrow e_6) & = & e_4, R_4, R_5, e_6 \end{array}$$

but there is no 2-connection from $(e_2, e_3) \leftrightarrow (e_7, e_4)$, because each path $(e_2 \leftrightarrow e_7)$ and $(e_3 \leftrightarrow e_8)$ would pass through R_8.

9.2 The Cut-point Lemma

Suppose $\mathcal{A}$ is a family of chords in the disc D. Let K be the cell complex constructed in Section **??** and suppose that K is 2-colored black and white. A pair of points X and Y on the boundary circle C are called *cut-points* for $\mathcal{A}$ if X and Y are distinct points on C, neither of which is the endpoint of an arc in $\mathcal{A}$. The cut-points may be in (the interior) of either a black interval or a white interval in C. A pair of cut-points, X and Y, separate the boundary circle C into two arcs, where $\widehat{XY}$ is the clockwise open interval from X to Y, $\widehat{YX}$ is the clockwise open arc from Y to X. The circle C is the disjoint union of X, $\widehat{XY}$, Y, and $\widehat{YX}$. Let X and Y be a pair of cut-points for $\mathcal{A}$. A chord in $\mathcal{A}$ is called *re-entrant* in $\widehat{XY}$ if both of its endpoints are in the open arc $\widehat{XY}$. Suppose that there are two sets $E = \{e_i\}$ and $F = \{f_i\}$, each with k black intervals, and $(e_1, \ldots, e_k, f_k, \ldots, f_1)$ are in circular order around C. A k-connection from E to F. is said to *respect* the cut-points X and Y if for each $1 \leq i \leq k$, e_i is in the open arc $\widehat{XY}$ and f_i is in the open arc $\widehat{YX}$. With these preliminaries, we make the following definitions.

- $\mathrm{m}(X, Y)$ = the maximum integer k such that there is a k-connection which respects the cut-points X and Y.
- $r(X, Y)$ = the number of re-entrant chords in $\widehat{XY}$.
- $n(X, Y)$ = the number of black intervals which are entirely within $\widehat{XY}$.

If it is necessary to indicate the family $\mathcal{A}$, the notations $m(X, Y; \mathcal{A})$, $r(X, Y; \mathcal{A})$ and $n(X, Y; \mathcal{A})$ will be used.

Lemma 9.1 Cut-point Lemma *Suppose $\mathcal{A}$ is a finite family of chords in the disc, and assume that $\mathcal{A}$ is lensless. Let X and Y be a pair of cut-points for $\mathcal{A}$. With $n(X, Y)$, $m(X, Y)$ and $r(X, Y)$ defined as above,*

$$m(X, Y) + r(X, Y) - n(X, Y) = 0$$

Example 9.1 Refer to Figure **??** with X and Y placed as shown. In this case, $n(X, Y) = 3$, $m(X, Y) = 1$ and $r(X, Y) = 2$. Lemma **??** implies that there can be no 2-connection respecting the cutpoints X and Y, and there must be at least one 1-connection. There are many 1-connections respecting the cut-points X and Y, for example,

$$(e_1 \leftrightarrow e_7) = e_1, \quad R_1, R_7, R_6, e_7$$

There are no 2-connections through the graph because every 1-connection respecting the cut-points X and Y passes through the region R_7.

Example 9.2 In Figure **??**, with X and Y' placed as shown, $n(X, Y') = 3$, $m(X, Y') = 2$ and $r(X, Y') = 1$. Lemma **??** implies that there can be no 3-connection respecting the cutpoints X and Y', and there must be at least one 2-connection. One 2-connection (e_1, e_3) to (e_7, e_4) which respects the cut-points X and Y' is given by:

$$\begin{aligned} (e_1 \leftrightarrow e_7) &= e_1, R_1, R_7, R_6, e_7 \\ (e_3 \leftrightarrow e_4) &= e_3, R_3, R_8, R_4, e_4 \end{aligned}$$

This is an example where *shortening* the arc XY to XY' has the effect of *increasing* the maximum size of a connection. There can be no 3-connection respecting the cutpoints X and Y', because any such connection would need to use R_8 twice.

Proof: (of Lemma **??**) The family $\mathcal{A}$ will be altered to another lensless family of chords $\mathcal{C}$ in two steps; each step leaves the sum of the terms in the formula of Lemma **??** unchanged. The first step alters $\mathcal{A}$ to another family $\mathcal{B}$ such that there are no intersections in $\mathcal{B}$, so that no chord in $\mathcal{B}$ intersects any other chord in $\mathcal{B}$. The second step alters $\mathcal{B}$ to a family $\mathcal{C}$ for which there are no re-entrant chords in $\widehat{XY}$ or $\widehat{YX}$. The two steps of this alteration are as follows.

Step I. If there are any intersections of chords in $\mathcal{A}$, let $\mathcal{M}$ be the set of chords in $\mathcal{A}$ that intersect at least one other chord in $\mathcal{A}$, and let $\mathcal{N}$ be the set $\mathcal{A} - \mathcal{M}$. If $\mathcal{M}$ is empty, go on to Step II. Otherwise, Lemma **??** applies to the family $\mathcal{M}$, and implies that there are at least three empty boundary triangles. At least one of the empty boundary triangles has a boundary arc (defined by the endpoints of $\mathcal{M}$), containing neither X nor Y. Thus there must be two chords, say f and g in $\mathcal{M}$, which intersect at a point w, which meet C at u and v respectively, so that Δuvw an empty boundary triangle with neither cut-point X nor Y in arc uv. Note: u, v may not be adjacent

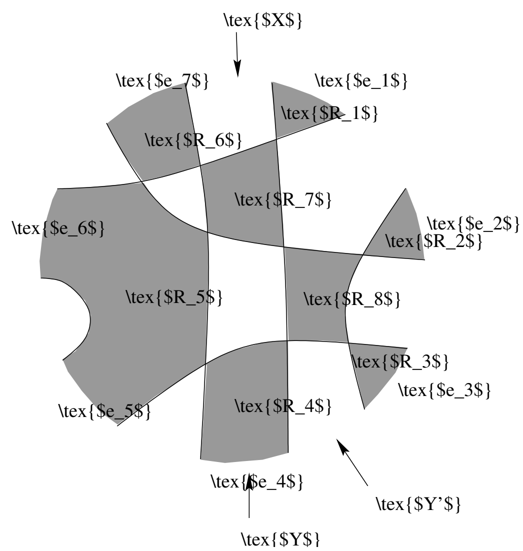


Figure 9-2:

endpoints of arcs of $\mathcal{A}$. The arc uv must be either in arc XY or arc YX; suppose the former. If uv is not an arc defined by the endpoints of $\mathcal{A}$, there must be a chord σ in $\mathcal{N}$ with both endpoints in arc uv. Let σ be a chord in $\mathcal{N}$, which, together with a portion of arc uv, encloses a region with segments of no other chords inside the region. Such a chord σ is necessarily re-entrant in arc XY. Removal of σ decreases $n(X,Y)$ by 1, decreases $r(X,Y)$ by 1, and leaves $m(X,Y)$ unchanged.

After removing a finite number of chords from $\mathcal{N}$, there will be at least one empty boundary triangle Δuvw in $\mathcal{A}$, with vertex w the intersection of chords f and g in $\mathcal{A}$ and such that u, v are adjacent endpoints of arcs of $\mathcal{A}$. An uncrossing of f and g will produce another family of chords $\mathcal{A}'$. We will show that each k-connection in $\mathcal{A}$ which respects the cut-points gives a k-connection $\mathcal{A}'$ which also respects the cut-points. First consider a 1-connection ($e \leftrightarrow f$). There are two cases, depending on whether uv is a black interval or a white interval.

(i) uv is a black interval, call it d. Refer to Figure **??**a. Suppose that R is the region in $\mathcal{A}$ bounded by uv, vw and wu, and suppose that S is the

(a) Arcs crossed (b) Arcs uncrossed

Figure 9-3: Black interval

.

(a) Crossed (b) Uncrossed

Figure 9-4: White interval

.

other black region in $\mathrm{bl}(K)$ with vertex w. The regions in $\mathrm{bl}(K')$ are labeled so that S' corresponds to the union of S and R Let

$$(e \leftrightarrow f) = e, B_1, B_2, \ldots, B_k, f$$

be any 1-connection in $\mathcal{A}$ that respects the cut-points X and Y. If S appears in the list, say as B_i, then take $B_i' = S'$ and there will be a connection in $\mathcal{A}'$ of the form

$$(d \leftrightarrow f) = d, B_i', B_{i+1}, \ldots, B_k, f$$

More generally, any k-connection $E \leftrightarrow F$ in $\mathcal{A}$ that respects the endpoints and uses the region S will give a k-connection in $\mathcal{A}'$ using S'. Any k-connection $E \leftrightarrow$F that does not use S is unaltered in $\mathcal{A}'$.

(ii) uv is a white interval. Refer to Figure **??**a. In this case, the two boundary intervals tu and vz on either side of uv are both black intervals. Consideration of the black regions R_1 and R_2 in the complexes K and K', shows that any connection in $\mathcal{A}$ starting from an interval in $\widehat{XY}$ will give a connection in $\mathcal{A}'$ also starting from an interval in the arc $\widehat{XY}$.

In either case, the maximum size k of a k-connection will be unchanged. Since each of the terms $m(X,Y)$, $n(X,Y)$ and $r(X,Y)$, in the sum is unchanged. A finite number of such uncrossings produces a family of chords $\mathcal{B}$, in which there are no crossings of chords. There may be some re-entrant chords as in Figure **??**.

Step II Since all the intersections have been removed, each re-entrant chord h does not intersect any other chord. If there are any re-entrant chords, there must be at least one which, together with the boundary circle, encloses a region containing no other segments of chords. The boundary interval of this region is in either arc XY or arc YX, suppose the former. Removal of such a region decreases $n(X,Y)$ by 1, decreases $r(X,Y)$ by 1, and leaves $m(X,Y)$ unchanged. The new family of chords has one less re-entrant chord. After a finite number of re-entrant chords have been removed, there will be a family of chords, $\mathcal{C}$, which has no crossings, and no re-entrant chords in $\widehat{XY}$ or $\widehat{YX}$. For this family,

$$n(X,Y) = m(X,Y)$$

This implies the formula for $\mathcal{A}$.

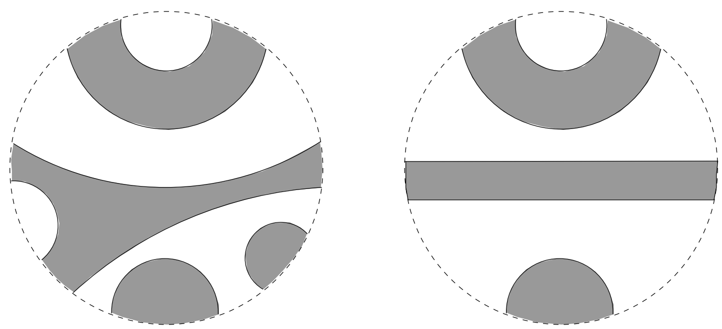


Figure 9-5:

Observation 9.1 Suppose $\mathcal{A}$ is a family of chords in the disc D, the cell complex K 2-colored black and white, and X and Y are cut points for $\mathcal{A}$ as in the discussion leading up to the cut-point Lemma **??**. Let L be a chord in the disc D from Y to X, which passes through no intersection point of $\mathcal{A}$, and which intersects each chord in $\mathcal{A}$ at most once. There is a "circle", C_1 consisting of X, $\widehat{XY}$, Y, and L in clockwise order. Let D_1 be the closed disc which C_1 bounds, and let $\mathcal{A}_1$ be the restriction of the chords in $\mathcal{A}$ to the disc D_1. Let K_1 be the complex in D_1 constructed from the family $\mathcal{A}_1$. K_1 inherits a 2-coloring from the 2-coloring of K. There are a finite number of intervals in C_1, each colored black or white. Similarly there is a "circle" C_2, a disc D_2, a family $\mathcal{A}_2$ and a complex K_2 inside C_2. Appearing in clockwise order around C_2, are Y, $\widehat{YX}$, X, and (the reverse of) L. Let k be the number of black intervals entirely within L. X and Y are a pair of cutpoints for the family $\mathcal{A}_1$. There are no re-entrant chords in L (considered as a subset of C_1). By Lemma **??**,

$$m(Y, X; \mathcal{A}_1) = k$$

Thus there is a connection $E \leftrightarrow F$ in $\mathrm{bl}(K_1)$, where $E = \{e_1, \ldots, e_k\}$ are the black intervals in L, and $F = \{f_1, \ldots, f_k\}$ are a set of black intervals in $\widehat{XY}$. Similarly

$$\mathrm{m}(X, Y; \mathcal{A}_2) = k$$

and there is a connection $E \leftrightarrow G$ in $\mathrm{bl}(K_2)$, where $G = \{g_1, \ldots, g_k\}$ are a set of black intervals in $\widehat{YX}$. Putting these two connections together, there is a connection $F \leftrightarrow G$ in $\mathrm{bl}(K)$. There can be no connection in $\mathrm{bl}(K)$ of size greater than k, because any connection must pass through the black intervals of L. Therefore

$$m(X, Y; \mathcal{A}) = k$$

9.3 Recovering a Medial Graph

Let $\mathcal{A}$ be a lensless family of chords in the disc D. The equivalence class of $\mathcal{A}$ may be recovered from the set of all k-connections of $\mathcal{A}$. What is actually needed from the set of all k-connections of $\mathcal{A}$ is the set of numbers $\{m(X, Y)\}$, as stated in the following Proposition.

Proposition 9.2 *Suppose that $\mathcal{A}$ is a lensless family of chords in the disc D. The equivalence class (under switches) of $\mathcal{A}$ may be calculated from the numbers $\{m(X, Y)\}$, as X and Y vary over the possible cut-points for $\mathcal{A}$.*

Proof: For each pair of cut-points X and Y, Lemma **??** implies that:

$$r(X, Y) = n(X, Y) - m(X, Y)$$

The numbers $r(X, Y)$ will be shown sufficient to determine the endpoints of the chords in $\mathcal{A}$. By Theorem **??**, this is sufficient to determine the equivalence class of $\mathcal{A}$. Suppose that there are n chords, the endpoints of which are $\{t_1, \ldots, t_{2n}\}$, numbered in circular order around C. The set $\{t_1, \ldots, t_{2n}\}$ is ordered by its indices. For each $1 \leq i < 2n$, place a point X_i in the open interval (t_{i-1}, t_i) on C; the convention is that $t_0 = t_{2n}$ and $X_0 = X_{2n}$. Then $\{X_1, X_2, \ldots, X_{2n}\}$ is also a set ordered by its indices. The points t_i and X_i appear in the following order around C.

$$t_0 < X_1 < t_1 < X_2 < \ldots t_{2n-1} < X_{2n} < t_{2n} = t_0$$

If it occurs, the point t_{2n+j} is to be identified with the point t_j; point X_{2n+j} is to be identified with the point X_j.

For each $i < j$, let $R(i, j) = r(X_i, X_j)$ be the number of re-entrant geodesics between the cut-points X_i and X_j.

(1) Let j be the first index greater than 1 for which $R(i,j) \neq R(2,j)$. Then t_1 and t_{j-1} are the endpoints of a chord in $\mathcal{A}$.

(2) Similarly, for each $i = 2, \ldots, 2n$, let j (which depends on i) be the first index after i for which $R(i,j) \neq R(i+1,j)$. Then t_i and t_{j-1} are the endpoints of a chord in $\mathcal{A}$.

This procedure locates the endpoints of each of the chords in $\mathcal{A}$. Each chord will occur twice; the second time the endpoints are in reversed order. Any family of chords constructed with these endpoints has the same endpoints as the original family $\mathcal{A}$, and so, by Theorem **??**, is in the same equivalence class.

9.4 Examples

Suppose $\Gamma = (G, \gamma)$ is a circular planar graph, for which G is critical as a graph. Proposition **??** shows that the ranks of the sub-determinants of the response matrix Λ are sufficient to calculate the numbers $m(X, Y)$, which are sufficient to determine the equivalence class of the medial graph, and hence the $Y - \triangle$ equivalence class of Γ.

Example 9.3 Figure **??** shows a graph G with 6 boundary nodes, 4 interior nodes, and 12 edges.

Suppose the conductivity of each edge e is $\gamma(e) = 1$, and let $\Gamma = (G, \gamma)$ denote the resistor network. The Kirchhoff matrix for Γ is:

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 3 \end{bmatrix}$$

The response matrix Λ is calculated as the Schur complement of K with respect to the 4×4 matrix in the lower right corner. The result is:

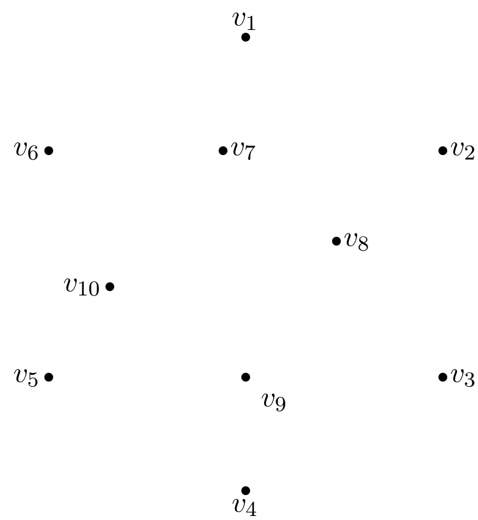


Figure 9-6: Graph G

$$\Lambda = \begin{bmatrix} 0.6203 & -0.1392 & -0.0380 & -0.0380 & -0.0127 & -0.3924 \\ -0.1392 & 1.5823 & -1.1139 & -0.1139 & -0.0380 & -0.1772 \\ -0.0380 & -1.1139 & 2.6962 & -1.3038 & -0.1013 & -0.1392 \\ -0.0380 & -0.1139 & -1.3038 & 1.6962 & -0.1013 & -0.1392 \\ -0.0127 & -0.0380 & -0.1013 & -0.1013 & 0.6329 & -0.3797 \\ -0.3924 & -0.1772 & -0.1392 & -0.1392 & -0.3797 & 1.2278 \end{bmatrix}$$

Now suppose that only the 6×6 response matrix Λ is given. The $Y - \triangle$ equivalence class of the graph G can be determined by finding the equivalence class of the medial graph $\mathcal{M}$, following the procedure outlined in Proposition **??**. The ranks of submatrices suitable for use in Proposition **??** can be computed from the matrix Λ. The following table lists the numbers $m(X_i, X_j)$, for all $1 \le i \le 12$, and all $1 \le j \le 22$. The number at row i, column j is the value of $m(X_i, X_j)$:

0	0	1	1	2	2	3	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	2	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	2	2	2	2	2	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	1	1	2	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	1	2	2	3	2	2	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	2	1	2	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	2	2	3	2	2	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	2	2	2	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	2	2	2	1	2	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	1	2	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	2	3	2	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	1	2	1	1	0

The next table lists the numbers $r(X_i, X_j)$, for all $1 \leq i \leq 12$, and all $1 \leq j \leq 22$. These numbers are computed by the formula of Proposition **??**:

$$r(X_i, X_j) = n(X_i, X_j) - m(X_i, X_j)$$

The number at row i, column j is the value of $R(i, j)$:

0	0	0	0	0	0	0	1	2	3	4	5	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	2	3	4	5	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	2	3	4	5	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	2	3	4	5	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	4	5	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	2	3	4	5	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	4	5	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	3	4	5	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	4	5

The pairing of the endpoints of the chords of $\mathcal{A}$ is found from the list of numbers $r(X_i, X_j, \mathcal{A})$. The calculation of the first six pairs goes as follows.

(1) $R(1, 8) = 1 \neq R(2, 8) = 0$, so there is a chord from t_1 to t_7.

(2) $R(2, 10) = 2 \neq R(3, 10) = 1$, so there is a chord from t_2 to t_9.

(3) $R(3,11) = 2 \neq R(4,11) = 1$, so there is a chord from t_3 to t_{10}.

(4) $R(4,9) = 1 \neq R(5,9) = 1$, so there is a chord from t_4 to t_8.

(5) $R(5,13) = 2 \neq R(6,13) = 1$, so there is a chord from t_5 to t_{12}.

(6) $R(6,12) = 2 \neq R(7,12) = 1$, so there is a chord from t_6 to t_{11}.

The pairing of the endpoints is: $(1,7)$, $(2,9)$, $(3,10)$, $(4,8)$, $(5,12)$, $(6,11)$; the pairing is repeated as $(7,1)$, $(9,2)$, $(10,3)$, $(8,4)$, $(12,5)$, $(11,6)$. Refer to Figure **??** one way of placing these chords in the disc is shown as $\mathcal{A}$. A graph H is then obtained by 2-coloring the regions, placing a vertex in each of the black regions, and placing edges as explained in Section **??**. The graph $\mathcal{M}$ shown in Figure **??** is the medial graph for G. The graphs $\mathcal{M}$ and $\mathcal{A}$ are equivalent (by switches), and the graphs G and H are $Y - \triangle$ equivalent. The switches that change $\mathcal{M}$ to $\mathcal{A}$ are the following.

(1) The three chords with endpoints (1,7), (3,10), and (5,12) form an empty triangle, and should be switched.

(2) The three chords with endpoints chord (2,9), (3,10), and (6,11) form an empty triangle, and should be switched.

The transformations that change G into H are the following.

(1) A $Y - \triangle$ transformation at vertex v_{10} of G.

(2) A $Y - \triangle$ transformation at vertex v_8 of G.

Example 9.4 The connections through the graph G (and through the graph G') can be found by calculating sub-determinants of Λ. For example,

$$\begin{aligned}
\det \Lambda(1,2,3;6,5,4) &= 0.0127 \\
\det \Lambda(2,3,4;5,6,1) &= 0.0000 \\
\det \Lambda(3,4,5;2,1,6) &= 0.0127 \\
\det \Lambda(3,4;5,6) &= 0.0000
\end{aligned}$$

(1) The statement that $\det \Lambda(1,2,3;6,5,4) \neq 0$ shows that there is a 3-connection from (v_1, v_2, v_3) to (v_6, v_5, v_4). This 3-connection in G is

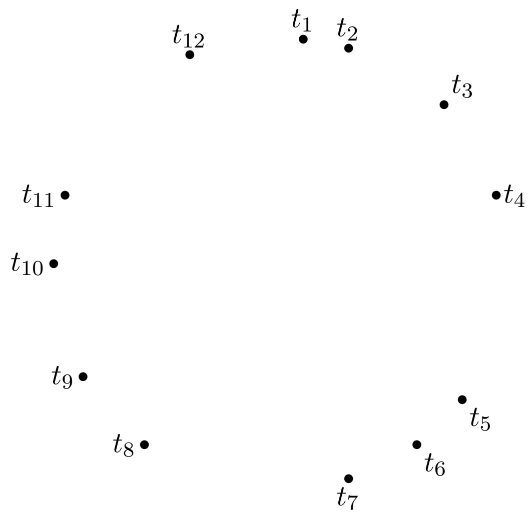


Figure 9-7: Placing the chords in $\mathcal{A}$

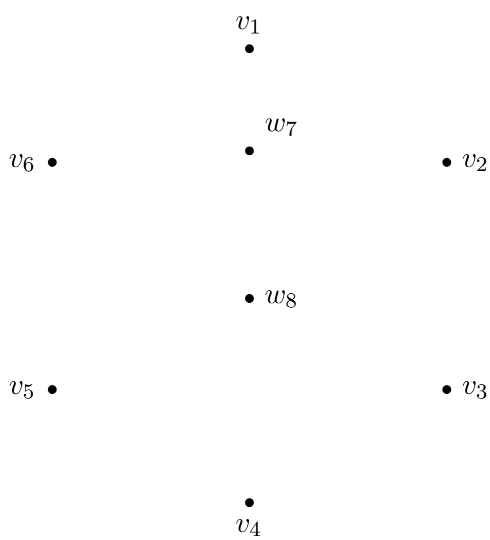


Figure 9-8: Graph H

$$\begin{aligned}
(v_1 \leftrightarrow v_6) &= v_1 v_7 v_6 \\
(v_2 \leftrightarrow v_5) &= v_2 v_8 v_9 v_{10} v_5 \\
(v_3 \leftrightarrow v_4) &= v_1 v_4
\end{aligned}$$

There is a corresponding 3-connection in G' from (w_1, w_2, w_3) to (w_6, w_5, w_4).

(2) The statement that $\det \Lambda(2,3,4;5,6,1) = 0$ shows that there is no 3-connection from (v_2, v_3, v_4) to (v_5, v_6, v_1).

(3) The statement that $\det \Lambda(3,4,5;2,1,6) \neq 0$ shows that there is a 3-connection from (v_3, v_4, v_5) to (v_2, v_1, v_6).

(4) The statement that $\det \Lambda(3,4;6,5) = 0$ shows that there is no 2-connection from (v_3, v_4) to (v_6, v_5).

Example 9.5 The connections through a graph can be examined from a slightly different point of view. Suppose the graph G of Figure **??** is given, and the medial graph $\mathcal{M}$ is constructed as shown in Figure **??**. The geodesics (listed by their endpoints) are:

$$(1,7), (2,9), (3,10), (4,8), (5,12), (6,11).$$

Points $t_1, \ldots, t_{12} = t_0$ and $X_1, \ldots, X_{12}$ are placed on the circle in clockwise sequence:

$$t_0 < X_1 < t_1 < X_2 < \ldots t_{2n-1} < X_{2n} < t_{2n} = t_0$$

(1) There are no re-entrant chords between the cutpoints X_1 and X_7, and there are three black intervals between X_1 and X_7, corresponding to the boundary nodes v_1, v_2, and v_3 in G. Lemma **??** implies that there must be a 3-connection from (v_1, v_2, v_3) through G to (v_6, v_5, v_4) and a corresponding 3-connection from (w_1, w_2, w_3) through G' to (w_6, w_5, w_4). In each case, there is only one such 3-connection which is listed above in Example **??**.

(2) There is one re-entrant chord between X_3 and X_9, which is the chord with endpoints (3,8). The number of black intervals between X_3 and X_9 is three. Lemma **??** implies that there cannot be a 3-connection from (v_2, v_3, v_4) to (v_1, v_6, v_5) through G, but there must be a 2-connection from a

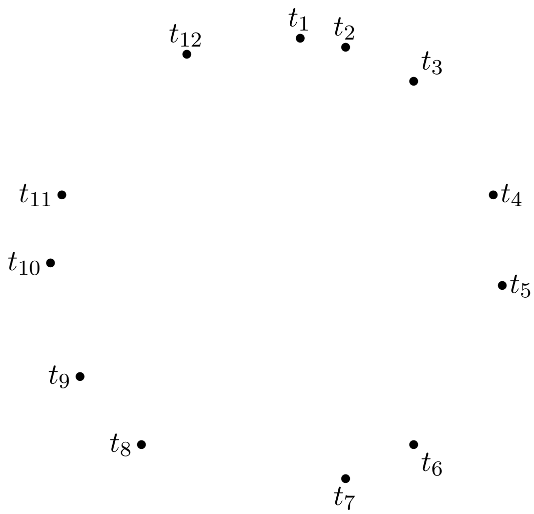


Figure 9-9: Medial graph $\mathcal{M}$

subset of (v_2, v_3, v_4) to a subset of (v_1, v_6, v_5). One such is (v_2, v_3) to (v_5, v_4). This connection, and the corresponding connection in G' are given by:

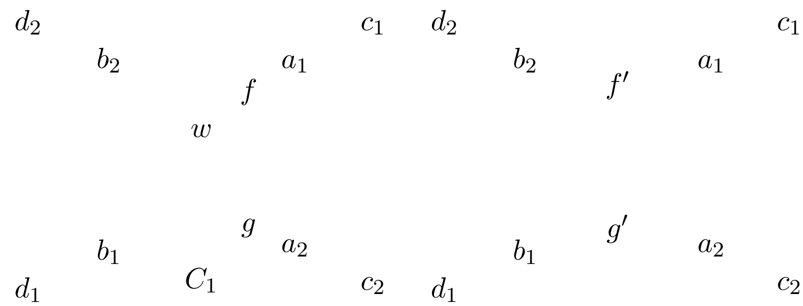
$$\begin{aligned} (v_2 \leftrightarrow v_5) &= v_2 v_8 v_9 v_{10} v_5 \qquad & (w_2 \leftrightarrow w_5) = w_2 w_7 w_5 \\ (v_3 \leftrightarrow v_4) &= v_3 v_4 \qquad & (w_3 \leftrightarrow w_4) = w_3 w_4 \end{aligned}$$

9.5 Critical Graphs

Proposition 9.3 *A circular planar graph G is critical if and only if its medial graph $\mathcal{M}$ is lensless.*

Proof: (1) If there were a lens in $\mathcal{M}$, then G would be $Y - \triangle$ equivalent to a graph G' which has a pair of edges in series or in parallel, or with an interior pendant or an interior loop. In each case an edge could be removed from G without breaking any connection, so G would not be critical.

(2) Suppose $\mathcal{M}$ has no lens. Let w be a vertex in $\mathcal{M}$ at which an uncrossing is made, and suppose f and g are the two chords which intersect at w. We must show that uncrossing f and g at w breaks some connection

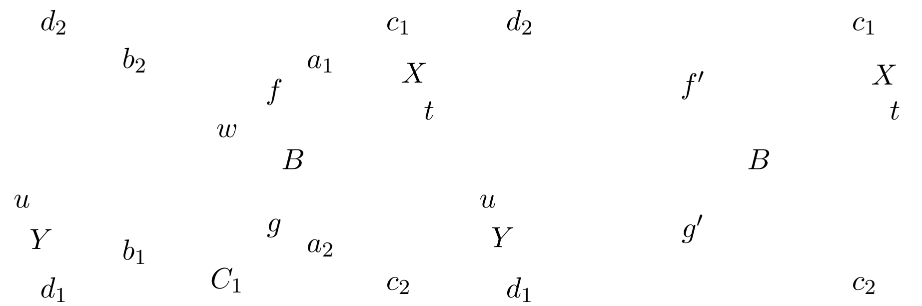


(a) Arcs crossing (b) Arcs uncrossed

Figure 9-10:

in bl(K). Let C_1 be a circle around w small enough so that no chords other than f or g go through C_1, as in Figure **??**. Suppose the endpoints of f are c_1 and d_1, and that f intersects C_1 at a_1 and b_1, so that c_1, a_1, b_1, d_1 appear in order along f. Similarly, the endpoints of g are c_2 and d_2, and g intersects C_1 at a_2 and b_2, so that c_2, a_2, b_2, d_2 appear in order along g, as shown in Figure **??**. After the uncrossing the points c_1, a_1, b_2 and d_2 will appear in order along f'; the points c_2, a_2, b_1 and d_1 will appear in order along g'.

Suppose the next endpoint after c_1 (in clockwise order) on C, of a chord in $\mathcal{M}$ is t and the next endpoint of a chord in $\mathcal{M}$ on C after d_1 is u. Place a point X in segment c_1t, and a point Y in segment d_1u. Let B be a chord (not in the family $\mathcal{M}$) joining X to Y. By placing B sufficiently close to chord f, we may assume that B intersects each chord of $\mathcal{M}$ at most once, and that going from X to Y, B crosses first g and then f within circle C_1 as shown in Figure 9-19a. After the uncrossing, we may assume that B does not intersect the chord f' or the chord g', as shown in Figure **??**. Starting from Y and going clockwise, the arc $\widehat{YX}$ followed by the chord B form a circle C_1 which bounds a disc D_1. This disc contains a family of chords $\mathcal{N}$, namely the restrictions of the chords of $\mathcal{M}$ to D_1. The points X and Y may be considered as a pair of cut-points for $\mathcal{M}$ as well as for $\mathcal{N}$. The notation $m(X, Y; \mathcal{M})$ $m(X, Y; \mathcal{N})$ will be used to distinguish the numbers



(a) Placing B (b) Uncrossing the arcs

Figure 9-11:

of connections for the families $\mathcal{M}$ and $\mathcal{N}$ respectively. By the placing of B, there are no re-entrant chords in B (considered as an arc of the circle C_1). Let $n(X,Y;\mathcal{N})$ be the number of black intervals entirely within B. Before the uncrossing,

$$m(Y,X;\mathcal{M}) = m(Y,X;\mathcal{N}) = m(X,Y;\mathcal{N})$$

After the uncrossing, the number of black intervals within B decreases by one. Thus the maximum size of a connection through $\mathcal{M}'$ respecting the cut-points X and Y will be at most $m(Y,X;\mathcal{M}) - 1$. This shows that a connection must be broken in passing from $\mathcal{M}$ to $\mathcal{M}'$. Consequently, a connection must be broken in passing from G to G'.

Recall that two circular planar graphs G and G' are electrically equivalent if G can be transformed into G' by a finite sequence of $Y - \Delta$ transformations and trivial modifications. If G is electrically equivalent to G', then for any conductivity γ on G there is a conductivity γ' on Γ' so that the response matrix for (G,γ) is the same as the response matrix for (G,γ'). One consequence of Theorem **??** is the following.

Corollary 9.4 *Any circular planar graph G is electrically equivalent to a critical graph G', which is unique to within $Y - \Delta$ equivalence. Any two critical circular planar graphs that are electrically equivalent have the same number of arcs.*

Theorem 9.5 *Suppose G is a circular planar graph which is critical. Then the $Y-\triangle$ equivalence class of G may be calculated from the set of connections through G.*

Proof: The medial graph $\mathcal{M} = \mathcal{M}(G)$ is lens-free, by Lemma **??**. By Proposition **??**, the equivalence class of $\mathcal{M}$ may be calculated from the set of its connections, which are the same as the connections in G. By Theorem **??**, the equivalence class of G is determined by the equivalence class of $\mathcal{M}$.

Theorem 9.6 *Suppose Λ is a matrix which satisfies conditions $(P1)$, $(P2)$, $(P3)$ of Section **??**. Then there is a circular planar graph G, and there is a conductivity γ on G so that the response matrix for the resistor network (G, γ) is Λ.*

Proof: The connections in G are determined by the subdeterminants in Λ. Theorem **??** applies to yield a critical graph G. Theorem **??** of Chapter **??** shows that there is a conductivity γ on G so that the response matrix for the resistor network (G, γ) is Λ.

Observation 9.2 Suppose Λ is a matrix which satisfies conditions $(P1)$, $(P2)$, $(P3)$ of Section **??**. A resistor network $\Gamma = (G, \gamma)$ whose response matrix is Λ can be found in two steps. The first step is to obtain a graph G. The second step is to calculate conductivities γ on G.

Step I. A family of chords $\mathcal{A}$ is obtainable from Λ, as shown by Proposition **??**. $\mathcal{A}$ gives rise to a complex K which is 2-colored as in Section **??**. The complex $\mathrm{bl}(K)$ gives rise to a graph G. The family $\mathcal{A}$ is lens-free, so G is critical. Figure **??** shows such a family $\mathcal{A}$. The chords in $\mathcal{A}$ are the solid lines (some with corners); the dashed line $L = XY$ is an auxiliary chord to be added later. The graph G consists of all the solid lines and dotted lines in Figure **??**.

Step II. By Lemma **??**, there must be at least one boundary spike or boundary edge in G. Suppose the former (the other case is similar). Observation **??** can be used to find the connection that is broken when the spike is contracted. By re-indexing if necessary, assume the boundary spike is $v_1 r$. Let $t_1 < v_1 < t_2$ be in clockwise order on the boundary circle and let $\sigma_1 = t_1 t_a$ and $\sigma_2 = t_2 t_a$ be the chords which intersect at the midpoint of

$v_1 r$. Place cutpoints X and Y on C, with $t_{2n} < X < t_1$ and $t_a < Y < t_{a+1}$ as shown in Figure **??**. Place a chord L joining X to Y not passing through any intersection points of $\mathcal{M}$, intersecting each chord in $\mathcal{M}$ at most once, and close enough to chord $t_1 t_a$ so that any chord which begins in $\widehat{t_1 t_a}$ and crosses σ_1, next crosses L before crossing any other chord of $\mathcal{M}$. The the uncrossing of σ_1 and σ_2 can be chosen so that the new chords are $\sigma_1' = (t_1 t_b)$ and $\sigma_2' = (t_2 t_a)$, and neither σ_1' nor σ_2' intersects L. Referring to Observation **??**, let k be the number of black intervals entirely within L. The number of black intervals entirely within L after the uncrossing is $k-1$. Thus there is a connection of size k which is broken when $v_1 r$ is contracted. This connection can be found as follows. There must be a submatrix of Λ of rank k whose rows correspond to vertices in $\widehat{t_1 t_a}$ and whose columns correspond to vertices in $\widehat{t_a t_1}$. This submatrix must contain a k by k submatrix $\Lambda(P;Q)$ with $\det \Lambda(P;Q) \neq 0$ and with $\det \Lambda'(P;Q) = 0$ where Λ' is the response matrix for the network Γ' which is Γ with the spike $v_1 r$ contracted.

Step III. The calculation of the conductance $\gamma(vr_3)$ will lead us to another explanation of the boundary spike formula (**??**) of Chapter **??**. (There is a similar explanation for the boundary edge formula.) According to Theorem **??**, there is a γ-harmonic function u on G, with

(1) $u(v_1) = 1$

(2) $u(v_i) = 0$, if $v_i \notin Q$

(3) $\phi_u(v_i) = 0$ if $v_i \in P$

The algorithm of Section **??** will apply if it can be shown that $u(r) = 0$. As in Observation **??**, let K_1 be the complex in the disc D_1, with boundary circle C_1 which is X, $\widehat{XY}$, Y, and L. This complex K_1 may be 2-colored so that the black regions are the nodes of a graph G_1 which is a subgraph of G.

An example of such a subgraph G_1 is shown in Figure **??**, where G_1 consists of the nodes and vertices below the dashed line XY. The subnetwork $\Gamma_1 = (G_1, \gamma_1)$, where γ_1 is the restriction of γ to G_1. In this example, $k = 3$, and the vertices of G_1 along the portion $L = YX$ of the boundary circle for G_1 are labeled r_1, r_2, and r_3. The boundary circle for G_1 passes through v_1, v_2, v_3, v_4, v_5, r_1, r_2 and r_3.

- The boundary spike $v_1 r_3$ in G is a boundary pendant in G_1.

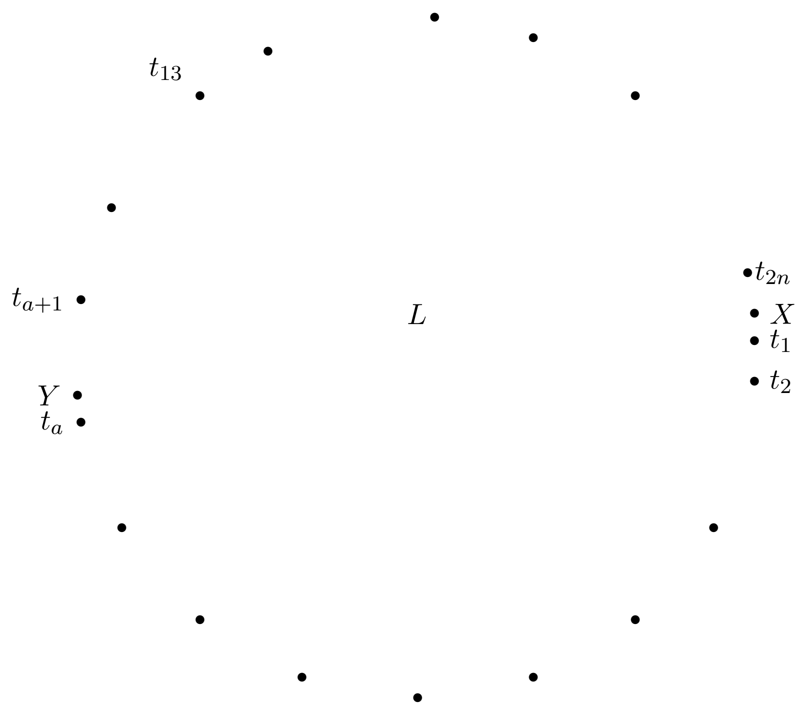


Figure 9-12: Family $\mathcal{A}$

By Theorem **??**, there is a unique γ-harmonic function w on G_1, with

(1) $w(v_1) = 1$

(2) $w(v_i) = 0$, if $v_i \notin R$

(3) $\phi_w(v_i) = 0$ if $v_i \in P$

The function which is zero on all of G_1 except $w(v_1) = 1$ is γ-harmonic, and satisfies these conditions. The γ-harmonic function u restricts to a *gamma*-harmonic function u_1 on G_1, which must be that same as w on G_1. Hence $u(r) = 0$. The algorithm of Section **??** applies, with the result that

$$\gamma(v_1 r) = -\Lambda(1; Q) \cdot \Lambda(P; Q)^{-1} \cdot \Lambda(P; 1) + \Lambda(1; 1)$$

which is the boundary spike formula (**??**) of Chapter **??**.

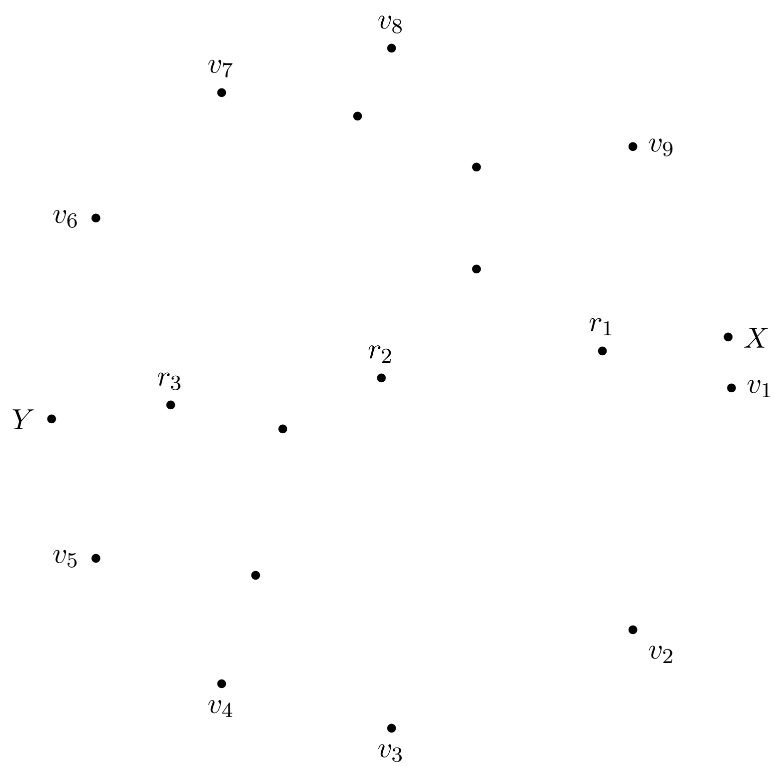


Figure 9-13: Network Γ and subnetwork Γ_1

Chapter 10

Layered Networks

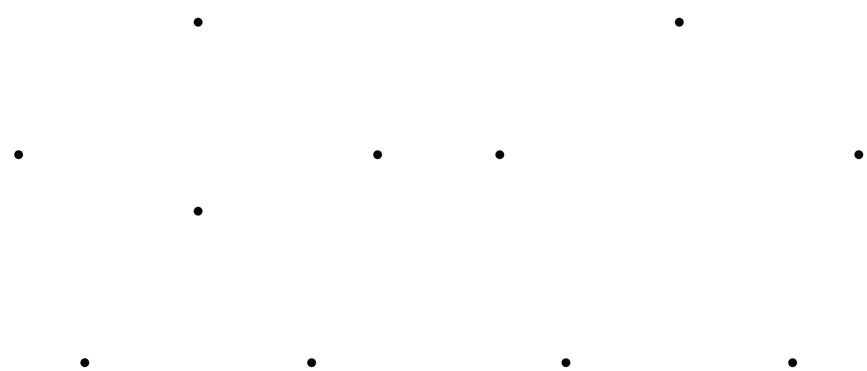
The material of this chapter is taken from David Ingerman's thesis, which is to appear in [**?**]. For circular planar networks that have a layered structure, it is possible to characterize the response matrix by its eigenvalues. The graphs of these networks will be referred to as discrete discs, denoted $D(n,l)$ and $D(n,l)^{\perp}$, where n is the number of rays, and l is the number of layers. To simplify the exposition, assume that $n = 2m+1$ is an odd integer. The figures in Figure **??** illustrate the different types. Figure **??**a shows a discrete disc $D(5,2)$ with 5 rays and 2 layers. The dual of $D(5,2)$ is the discrete disc $D(5,2)^{\perp}$ in Figure **??**b. The vertices are located at the points with polar coordinates (r_j, θ_k) where

(1) The r_j are equally spaced radii, with $r_0 = 0$.

(2) $\theta_k = \dfrac{2\pi k}{n}$.

The edges are the segments of radial lines and the circular arcs between adjacent vertices. The *layers* of the discs $D(n,l)$ and $D(n,l)^{\perp}$ are the minimal subsets of edges invariant under rotations of the graph through angle $\dfrac{2\pi}{n}$. Each layer consists of n edges. A *layered network* is a discrete disc with a conductivity function that is constant on layers. Thus the conductivity is given by l positive real numbers.

Each of the discrete discs $D(5,2)$ and $D(5,2)^{\perp}$ in Figure **??** has 5 rays and 2 layers. The conductivity is given by two real numbers. Figure **??** shows a discrete disc $D(11,5)$ with 11 rays and 5 layers. The conductivity is given by five real numbers.



(a) $D(5,2)$

(b) $D(5,2)^{\perp}$

Figure 10-1: Discrete discs

It is convenient to consider the boundary functions as functions of the angle θ and the potentials as functions of polar coordinates (r, θ). Thus $e^{ik\theta}$ will represent the boundary function with values at the boundary nodes

$$\phi_k(\theta_j) = e^{\frac{2ikj\pi}{n}} \text{ for } j = 1, \ldots, n$$

where $\theta_j = \dfrac{2\pi j}{n}$.

Lemma 10.1 *The solution to the Dirichlet problem on a discrete layered disc with boundary values given by the function $e^{ik\theta}$ is of the form*

$$u_k = a_k(r) e^{ik\theta}$$

Proof: Let u_k be a potential which, when restricted to the boundary, is $e^{ik\theta}$. Since the conductivity is constant on layers, the function

$$v(r, \theta) = u_k \left(r, \theta + \frac{2\pi}{n} \right) - u_k(r, \theta)$$

is also a potential. When restricted to the boundary, v is the function

$$v|_{\partial D_n} = e^{ik\theta} \left(e^{\frac{i2\pi k}{n}} - 1 \right)$$

By the uniqueness of the solution to the Dirichlet problem,

$$u_k(r,\theta)\left(e^{\frac{i2\pi k}{n}}-1\right)=u_k\left(r,\theta+\frac{2\pi}{n}\right)-u_k(r,\theta)$$

Hence

$$u_k(r,\theta)\left(e^{\frac{i2\pi k}{n}}\right)=u_k\left(r,\theta+\frac{2\pi}{n}\right)$$

and letting $\theta=\theta_1=\dfrac{2\pi}{n}$, we see that

$$u_k(r,\theta_2)=u_k(r,\theta_1)e^{-ik\theta_1}\cdot e^{ik\theta_2}$$

Similarly,

$$u_k(r,\theta_j)=u_k(r,\theta_j)e^{-ik\theta_1}\cdot e^{ik\theta_j}$$

Letting $a_k(r)=u_k(r,\theta_1)e^{ik\theta_1}$, we see that

$$u_k(r,\theta)=a_k(r)e^{ik\theta}.$$

Λ can be compared to a discretization of the differential operator $\dfrac{d^2}{d\theta^2}$ considered as an operator on boundary functions on the unit disc. The finite difference approximation to $-\dfrac{d^2}{d\theta^2}$ can be represented by the matrix:

$$\left[-\frac{d^2}{d\theta^2}\right]=\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & . & . & . & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & . & . & . & 0 & -1 & 2 \end{bmatrix}$$

This operator acts on the space of boundary functions defined on ∂D_n, $n=2m+1$. A complete set of eigenvectors is given by:

$$\{e^{ik\theta}:k=-m,\ldots,-1,0,1,\ldots,m\}$$

with eigenvalues

$$|e^{i\frac{2\pi k}{n}}-1|^2=\left(2\sin\frac{\pi k}{n}\right)^2$$

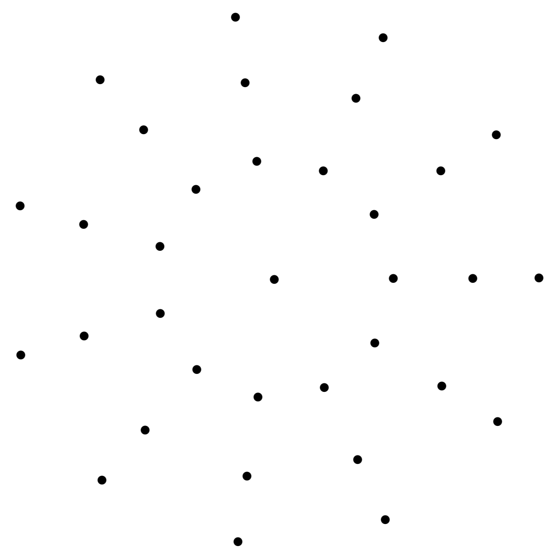


Figure 10-2: Discrete disc $D(11, 5)$

This follows from

$$
\begin{aligned}
2e^{ik\theta_j} - e^{ik\theta_{j+1}} - e^{ik\theta_{j-1}} &= e^{ik\theta_j}\left[2 - e^{\frac{ik2\pi}{n}} - e^{-\frac{ik2\pi}{n}}\right] \\
&= e^{ik\theta_j}\left[2 - 2\cos(\frac{2\pi k}{n})\right] \\
&= e^{ik\theta_j} \cdot 4\sin^2(\frac{\pi k}{n})
\end{aligned}
$$

Define

$$\omega_k^{(n)} = \omega_{-k}^{(n)} = |e^{\frac{i2\pi k}{n}} - 1|$$

The eigenvalues of Λ will now be computed. For simplicity, consider the case of $D(n, l)$, with l odd. Let $h = \dfrac{l+1}{2}$ and let $\{\sigma_1, \mu_1, \sigma_2, \ldots, \mu_{h-1}, \sigma_h\}$ be the conductivities on the layers of $D(n, l)$ outward starting from the origin. For example, the discrete disc $D(11, 5)$ in Figure **??** has 11 rays and 5 layers.

The conductivity is given by five real numbers, $\sigma_1, \mu_1, \sigma_2, \mu_2, \sigma_3$. Let

$$u_k(r,\theta) = a_k(r)e^{ik\theta}$$

be the solution of the Dirichlet problem with boundary values $e^{ik\theta}$. Let $k \neq 0$. For u_k to be well-defined at $r = 0$, we must have $a_k(0) = 0$. Since the boundary values are $e^{ik\theta}$, we must have $a_k(1) = 1$. Kirchhoff's Law at the interior nodes implies that

$$\sigma_j(a_k(r_j) - a_k(r_{j-1})) + \sigma_{j+1}(a_k(r_j) - a_k(r_{j+1})) + \mu_j \cdot a_k(r_j) \cdot (\omega_k^{(n)})^2 = 0$$

Let λ be a real parameter, and $P_0(\lambda), \ldots, P_h(\lambda)$ denote a sequence of functions. It will be convenient to think of the functions $P_j(\lambda)$ as functions of r_j, so we write $P_j(\lambda) = P(\lambda, r_j)$. This sequence of functions is defined by the following relations.

$$P(\lambda, 0) = P_0(\lambda) = 0$$
$$P(\lambda, 1) = P_h(\lambda) = 1$$

For $0 < j < h$,

$$\sigma_j(P_j(\lambda) - P_{j-1}(\lambda)) + \sigma_{j+1}(P_j(\lambda) - P_{j+1}(\lambda)) + \mu_j \cdot \lambda^2 \cdot P_j(\lambda) = 0$$

This is a linear system of equations that is analogous to the Kirchhoff equations (it is that system in disguise). The matrix of the system has the form:

$$\begin{bmatrix} \sigma_1 + \sigma_2 + \lambda^2\mu_1 & -\sigma_2 & 0 & \cdot & 0 \\ -\sigma_2 & \sigma_2 + \sigma_3 + \lambda^2\mu_2 & -\sigma_3 & \cdot & 0 \\ 0 & -\sigma_3 & \sigma_3 + \sigma_4 + \lambda^2\mu_3 & \cdot & 0 \\ & & & & \\ 0 & 0 & 0 & \cdot & \sigma_{h-1} + \sigma_h + \lambda^2\mu_{h-1} \end{bmatrix}$$

which is $L + D$, where

$$L = \begin{bmatrix} \sigma_1 + \sigma_2 & -\sigma_2 & 0 & \cdot & \cdot & 0 \\ -\sigma_2 & \sigma_2 + \sigma_3 & -\sigma_3 & \cdot & \cdot & 0 \\ 0 & -\sigma_3 & \sigma_3 + \sigma_4 & \cdot & \cdot & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \sigma_{h-1} + \sigma_h \end{bmatrix}$$

and $D = \text{Diag}\,[\lambda^2\mu_1, \lambda^2\mu_2, \ldots, \lambda^2\mu_{h-1}]$. L is the interior matrix of a linear network of resistors with conductivities $\sigma_1, \sigma_2, \ldots, \sigma_h$, and is positive definite. D is positive definite, so $L+D$ is also positive definite. Thus there is a unique solution to this linear system of equations. The functions $P_j(\lambda)$ are rational functions of λ. The eigenvalue $\lambda_k^{(n)}$ corresponding to the eigenvector $e^{ik\theta}$ is given by

$$\begin{aligned}
\lambda_k^{(n)} &= \sigma_h(a_k(1) - a_k(r_{h-1})) \\
&= \sigma_h(P(\omega_k^{(n)}, r_h) - P(\omega_k^{(n)}, r_{h-1})) \\
&= \sigma_h(P_h(\omega_k^{(n)}) - P_{h-1}(\omega_k^{(n)}))
\end{aligned}$$

The functions $P_j(\lambda)$ depend on the number of layers and the conductivities μ and σ on the layers, but not on k or n. For $j = 1, \ldots, h$, let

$$\begin{aligned}
Q_j(\lambda) = Q(\lambda, r_j) &= \sigma_j(P(\lambda, r_j) - P(\lambda, r_{j-1})) \\
&= \sigma_j(P_j(\lambda) - P_{j-1}(\lambda))
\end{aligned}$$

Then

$$\lambda_k^{(n)} = Q(\omega_k^{(n)}, 1) = \frac{Q(\omega_k^{(n)}, 1)}{P(\omega_k^{(n)}, 1)}$$

and

$$\begin{aligned}
P_j &= P_{j-1}(\lambda) + \frac{1}{\sigma_j} Q_j(\lambda) \\
Q_j &= Q_{j-1}(\lambda) + \mu_{j-1}\lambda^2 P_{j-1}(\lambda)
\end{aligned}$$

So

$$\begin{aligned}
\frac{Q_j}{P_j} &= \frac{Q_j}{P_{j-1}(\lambda) + \frac{1}{\sigma_j} Q_j(\lambda)} \\
&= \frac{1}{\frac{1}{\sigma_j} + \frac{P_{j-1}}{Q_j}} \\
&= \frac{1}{\frac{1}{\sigma_j} + \frac{P_{j-1}}{Q_{j-1} + \mu_{j-1}\lambda^2 P_{j-1}(\lambda)}} \\
&= \frac{1}{\frac{1}{\sigma_j} + \frac{1}{\mu_{j-1}\lambda^2 + \frac{Q_{j-1}(\lambda)}{P_{j-1}(\lambda)}}}
\end{aligned}$$

Continuing in this way, we get a continued fraction

$$R(\lambda) = \cfrac{1}{\frac{1}{\sigma_h} + \cfrac{1}{\mu_{h-1}\lambda^2 + \cfrac{1}{\sigma_{h-1} + \dots \cfrac{1}{\frac{1}{\sigma_2} + \frac{1}{\mu_1\lambda^2+\sigma_1}}}}}$$

Since $Q_1(\lambda) = \sigma_1(P_1(\lambda) - P_0(\lambda)) = P_1(\lambda)$ the eigenvalues $\lambda_k^{(n)}$ of Λ_Γ are given by

$$\lambda_k^{(n)} = R(\omega_k^{(n)})$$

By introducing the function

$$\beta(\lambda) = \frac{R(\lambda)}{\lambda} = \cfrac{1}{\frac{\lambda}{\sigma_h} + \cfrac{1}{\mu_{h-1}\lambda + \dots \cfrac{1}{\frac{\lambda}{\sigma_2} + \cfrac{1}{\mu_1\lambda + \frac{1}{\frac{\sigma_1}{\lambda}}}}}}$$

we can write

$$\lambda_k^{(n)} = \omega_k^{(n)}\beta(\omega_k^{(n)})$$

If β is considered to be a function of a complex variable λ, then β has the following properties:

(1) β is rational

(2) If $\Re(\lambda) > 0$, then $\Re(\beta(\lambda)) > 0$

(3) β is real

(4) $\beta(-\overline{\lambda}) = -\overline{\beta(\lambda)}$; that is, β is <u>para - odd</u>.

These four properties characterize continued fractions of this form; see [**?**]. Let $\mathcal{B}$ be the class of functions satisfying (1), (2), (3), and (4). The following theorem is due to David Ingerman.

Theorem 10.2 *Let $n = 2m+1$. A linear map $\Lambda : R^n \to R^n$ is the response map of a discrete layered disc D_n if and only if Λ is diagonal in the orthogonal basis*

$$\{e^{ik\theta}|_{\partial D_n} : k = -m, \ldots, 0, \ldots, m\}$$

$\Lambda 1 = 0$, *and there is a function $\beta \in \mathcal{B}$ such that for $k = 1, \ldots, m$,*

$$\Lambda e^{\pm ik\theta}|_{\partial D_n} = \omega_k^{(n)} \beta(\omega_k^{(n)}) \cdot e^{\pm ik\theta}|_{\partial D_n}$$

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