A Table of Isoperimetric Ratios

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For a simple closed plane curve, the isoperimetric ratio is

\[ \frac{P^2}{A} \]

where \( P \) is the length of the curve, and \( A \) is the area enclosed by the curve. This ratio is a dimensionless constant that is invariant under scaling: changing the size of the shape does not change this ratio.

The table below gives this ratio for a variety of shapes.

One can prove that the circle is the plane curve with the smallest (and hence, smallest possible) isoperimetric ratio, and so the table begins with the circle.
circle $\quad 4\pi \approx 12.56637061$

regular 11-gon $\quad 44\tan \frac{\pi}{11} \approx 12.91956568$

square with quarter circle rounded corners (radius of corners is one-third of overall width) $\quad \frac{(4+2\pi)^2}{9+\pi} \approx 12.98810988$

regular 10-gon $\quad 40\tan \frac{\pi}{10} \approx 12.99678784$

regular 9-gon $\quad 36\tan \frac{\pi}{9} \approx 13.10292843$

three-quarter truncated circular arc $\quad \approx 13.14170265$

regular octagon (equivalent to square augmented by four optimal isosceles triangles) $\quad 32\tan \frac{\pi}{8} \approx 13.25483399$

square with hacked off corners (horizontal and vertical edges are one-third overall width) $\quad \frac{(4+4\sqrt{2})^2}{7} \approx 13.32211914$
ellipse, 3:2 \[\approx 13.35374554\]

regular septagon \[28 \tan \frac{\pi}{7} \approx 13.48408932\]

cardiod \[\frac{128}{3\pi} \approx 13.58122181\]

semicircle with optimal adjoined isosceles triangle \[\approx 13.71361987\]

square with three optimal adjoined isosceles triangles \[\approx 13.75752837\]

16 pointed star \[\approx 13.77927647\]

equilaterial triangle with three adjoined semicircles \[\approx 13.78342300\]

regular hexagon \[\frac{24}{\sqrt{3}} \approx 13.85640646\]

semi-circle with adjoined right isosceles triangle \[\frac{(2\sqrt{2}+\pi)^2}{\frac{2}{\pi}+1} \approx 13.86385058\]
15 pointed star \[ \approx 13.96032741 \]
equilateral triangle with two adjoined semicircles \[ \frac{4(1+\pi)^2}{\pi+\sqrt{3}} \approx 14.07800126 \]
14 pointed star \[ \approx 14.18654408 \]
semicircle with optimal adjoined rectangle \[ \frac{(4+\pi)^2}{2+\frac{\pi}{2}} \approx 14.2831853071 \]
square with two optimal adjoined isosceles triangles \[ \approx 14.35072219 \]
13 pointed star \[ \approx 14.47485646 \]
regular pentagon \[ \frac{100}{\sqrt{25+10\sqrt{5}}} \approx 14.53085056 \]
square with two adjoined semicircles \[ \frac{(2+\pi)^2}{1+\pi} \approx 14.80676722 \]
12 pointed star \[ \approx 14.85125168 \]
ellipse, 2:1

\[ \approx 14.93924249 \]

square with three adjoined semicircles

\[ \frac{(1 + \frac{3}{2})^2}{1 + \frac{3}{2}\pi} \approx 14.98160283 \]

rectangle, 2:1, with four adjoined semicircles

\[ \frac{36\pi^2}{8\pi + 16} \approx 14.98676855 \]

square with adjoined semicircle

\[ \frac{(\pi + 6)^2}{\frac{5}{2}\pi + 4} \approx 15.00121550 \]

square with optimal adjoined isosceles triangle

\[ \approx 15.07344594 \]

square with single optimally rounded corner (radius is one half the side length)

\[ 12 + \pi \approx 15.14159265 \]

square with four adjoined semicircles

\[ \frac{4\pi^2}{1 + \frac{\pi}{4}} \approx 15.35649370 \]

11 pointed star

\[ \approx 15.35751730 \]

equilateral triangle with adjoined semicircle

\[ \frac{(4 + \pi)^2}{\frac{\pi}{2} + \sqrt{3}} \approx 15.44193344 \]
square

circular sector, two radians

10 pointed star ≈ 16.06491327

quarter circular sector

semicircle augmented by two semicircles, I

semicircle

equilateral triangle with adjoined optimal rectangle

9 pointed star

rectangle, 2:1

≈ 16.23455083

\frac{4(2+\frac{\pi}{2})^{2}}{\pi} ≈ 16.82966439

\frac{16}{3} \pi ≈ 16.75516081

\frac{2(2+\pi)^{2}}{\pi} ≈ 16.82966439

4(6 - \sqrt{3}) ≈ 17.07179676

≈ 17.10465828

≈ 17.07179676

18
circular sector, one radian

rectangle, 2:1, with two adjoined semicircles

8 pointed star

isosceles right triangle with optimal adjoined rectangle \((1 : \frac{\sqrt{2}}{2})\)

isosceles right triangle with adjoined square

equilateral triangle

rectangle, 3:1

7 pointed star

isosceles triangle, 2:1 sides, with adjoined semicircle
isosceles right triangle $2(2 + \sqrt{2})^2 \approx 23.31370849$

square with two adjoined isosceles right triangles $2(2 + \sqrt{2})^2 \approx 23.31370849$

square with four adjoined equilateral triangles $\frac{64}{1 + \sqrt{3}} \approx 23.42562584$

rectangle, 4:1 $25$

semicircle augmented with two semicircles, II $8\pi \approx 25.13274122$

right triangle, 30-60-90 $12 + \frac{24}{\sqrt{3}} \approx 25.85640646$

square with three adjoined right isosceles triangles $\frac{68 + 48\sqrt{2}}{5} \approx 27.17645019$

right triangle, 2:1 legs $(3 + \sqrt{5})^2 = 14 + 6\sqrt{5} \approx 27.41640786$

6 pointed star (Koch curve, stage 1) $\sqrt{708} \approx 27.71281292$
5-square cross
\[
\frac{144}{5} = 28.8
\]

rectangle, 5:1
\[
\frac{144}{5} = 28.8
\]

isosceles triangle, 2:1 sides
\[
\frac{50}{\sqrt{3}} \approx 28.86751345
\]

13-square diamond
\[
\frac{400}{13} \approx 30.76923076
\]

5-square cross with four adjoined semicircles
\[
\frac{8(4+\pi)^2}{\pi+10} \approx 31.04789318
\]

square with four adjoined isosceles right triangles
\[
16 + \frac{12}{3} \sqrt{2} \approx 31.08494466
\]

25-square diamond
\[
\frac{784}{25} = 31.36
\]

rectangle, 6:1
\[
\frac{98}{3} = 32.6
\]

isosceles triangle, 3:1 sides
\[
\frac{98}{\sqrt{8}} \approx 34.64823227
\]
rectangle, 7:1

\[
\frac{256}{7} \approx 36.57142857
\]

rectangle, 8:1

\[
\frac{81}{2} = 40.5
\]

Koch curve, stage 3

\[
\frac{384}{\sqrt[3]{3}} \approx 44.34050067
\]

9-square cross

\[
\frac{400}{9} = 44.4
\]

rectangle, 9:1

\[
\frac{400}{9} = 44.4
\]

5 pointed star

\[
20\sqrt{10} - 2\sqrt{5} \approx 47.02282018
\]

semicircle augmented by two semicircles, III

\[
16\pi \approx 50.26548245
\]

Koch curve, stage 4

\[
\frac{6144}{47\sqrt{3}} \approx 75.47319263
\]
General formulas

Rectangles
For a rectangle with aspect ratio $m$ (i.e., the length of a long side is $m$ times the length of a short side), the ratio is
$$\frac{(2m+2)^2}{m} = 4m + 8 + \frac{4}{m}.$$  

Regular polygons
For a regular polygon with $n$ sides, the ratio is $4n \tan \frac{\pi}{n}$.

Circular sectors
For a circular sector with angle $\theta$, the ratio is
$$\frac{2(2+\theta)^2}{\theta}.$$  
This has a minimum of 16 at $\theta = 2$.

Polygonal Stars
For an $n$-sided polygonal star with equal side lengths, like the one shown, with inner radius $r$ and outer radius $R$, the area is $\frac{1}{2}nrR \sin \frac{2\pi}{n}$ and the perimeter is $n\sqrt{r^2 + R^2 - 2rR \cos \frac{2\pi}{n}}$ so the isoperimetric ratio is
$$\frac{2n(r^2 + R^2 - 2rR \cos \frac{2\pi}{n})}{rR \sin \frac{2\pi}{n}}.$$ 

A special case of this are stars based on a regular polygon with $m \geq 5$ sides by extending the sides of the polygon until they intersect. In this case, we have $n = 2m$ and $R = r(\cos \frac{\pi}{m} + \sin \frac{\pi}{m} \tan \frac{2\pi}{m})$. In the table, such stars are called $n$-pointed stars.

$n$-square diamond/cross
The ratio is
$$\frac{(8m+12)^2}{2m^2 + 6m + 5}$$
where $m$ is the order of the diamond ($m = 0$ is the 5-square cross, $m = 1$ is the 13-square diamond, etc.)
This approaches 32 from below as $m$ tends to infinity.

Koch curve
The ratio is
$$\frac{180 \left(\frac{4}{9}\right)^{2n}}{\sqrt{3}(8 - 3 \left(\frac{4}{9}\right)^n)}$$
for the $n$-th iteration (i.e., $n = 0$ is an equilateral triangle, $n = 1$ is a six-pointed star, etc.)