A Table of Isoperimetric Ratios

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For a simple closed plane curve, the isoperimetric ratio is
\[
\frac{P^2}{A}
\]
where \( P \) is the length of the curve, and \( A \) is the area enclosed by the curve. This ratio is a dimensionless constant that is invariant under scaling: changing the size of the shape does not change this ratio.

The table below gives this ratio for a variety of shapes.

One can prove that the circle is the plane curve with the smallest (and hence, smallest possible) isoperimetric ratio, and so the table begins with the circle.

I started this table as a result of a homework problem in our precalculus class at the University of Washington. The problem asks the student to consider a situation in which we have a piece of wire of known length; we want to cut the wire and bend one resulting piece into a circle and the other piece into a square. The question is: where should we cut the wire so that the area of the circle plus the area of the square is minimized?

When writing exam questions of a similar flavor, the relative “efficiencies” of different planar shapes come under consideration.

For any wire length, if the two shapes have isoperimetric ratios \( r_1 \) and \( r_2 \), then the wire should be cut into two pieces with lengths in the ratio \( r_1 : r_2 \) if we wish to minimize the total area.

Cited by

This table is cited in the following paper:
circle \hspace{2cm} 4\pi \approx 12.56637061

regular 12-gon \hspace{2cm} 96 - 48\sqrt{3} \approx 12.86156124

regular 11-gon \hspace{2cm} 44 \tan \frac{\pi}{11} \approx 12.91956568

super ellipse \hspace{2cm} \approx 12.87612775

\|x|^3 + |y|^3 = 1

square with quarter circle rounded corners \hspace{2cm} \frac{(4 + 2\pi)^2}{5 + \pi} \approx 12.98810988

(radius of corners is one-third of overall width)

regular 10-gon \hspace{2cm} 40 \tan \frac{\pi}{10} \approx 12.99678784

regular 9-gon \hspace{2cm} 36 \tan \frac{\pi}{9} \approx 13.10292843
optimum region bounded by symmetric parabolic arcs (parabolas are approximately \( y = 0.97300151x^2 \) on the interval \((-0.97300151, 0.97300151)\) \(\approx 13.11934320\)

three-quarter truncated circular arc \(\approx 13.14170265\)

regular octagon (equivalent to square augmented by four optimal isosceles triangles) \(32(\sqrt{2} - 1) \approx 13.25483399\)

super ellipse \(|x|^4 + |y|^4 = 1\) \(\approx 13.28104122\)

square with hacked off corners (horizontal and vertical edges are one-third overall width) \(\frac{(4 + 4\sqrt{2})^2}{7} \approx 13.32211914\)

ellipse, 3:2 \(\approx 13.35374554\)

regular septagon \(28\tan\frac{\pi}{7} \approx 13.48408932\)
cardioid

3

\frac{128}{3\pi} \approx 13.5812181

super ellipse

|\frac{x}{5}| + |\frac{y}{5}| = 1

\approx 13.61890282

semicircle with optimal
adjoined isosceles triangle

\approx 13.71361987

square with three optimal
adjoined isosceles triangles

\approx 13.75752837

16 pointed star

\approx 13.77927647

equilaterial triangle

with three adjoined
semicircles

\approx 13.78342300

regular hexagon

\frac{24}{\sqrt{3}} \approx 13.85640646

semicircle with adj-
joined right isosceles triangle

\frac{2(2\sqrt{2} + \pi)^2}{\pi + 2} \approx 13.86385058
super ellipse
\(|x|^6 + |y|^6 = 1\)  
\[\approx 13.8994096\]

15 pointed star  
\[\approx 13.96032741\]

Reuleaux triangle  
\[
\frac{2\pi^2}{\pi - \sqrt{3}} \approx 14.00398921
\]

equilaterial triangle with two adjoined semicircles  
\[
\frac{4(1 + \pi^2)}{\pi + \sqrt{3}} \approx 14.07800126
\]

14 pointed star  
\[\approx 14.18654408\]

semicircle with optimal adjoined rectangle  
\[8 + 2\pi \approx 14.2831853071\]

square with two equally, optimally rounded corners (radii are half the side length)  
\[8 + 2\pi \approx 14.2831853071\]

square with two optimal adjoined isosceles triangles  
\[\approx 14.35072219\]
13 pointed star
\[ \approx 14.47485646 \]

regular pentagon
\[ \frac{100}{\sqrt{25 + 10\sqrt{5}}} \approx 14.53085056 \]

optimal region bounded by \( \frac{1}{x^2 + 1} \), the \( x \)-axis, and \( x = \pm a \), where \( a \approx 0.5181167 \)
\[ \approx 14.695335322 \]

square with two adjoined semicircles
\[ \frac{4(2 + \pi)^2}{4 + \pi} \approx 14.80676722 \]

12 pointed star
\[ \approx 14.85125168 \]

ellipse, 2:1
\[ \approx 14.93924249 \]

square with three adjoined semicircles
\[ \frac{2(2 + 3\pi)^2}{8 + 3\pi} \approx 14.98160283 \]

rectangle, 2:1, with four adjoined semicircles
\[ \frac{36\pi^2}{8 + 5\pi} \approx 14.98676855 \]
square with adjoined semicircle

\[
\frac{2(6 + \pi)^2}{8 + \pi} \approx 15.00121550
\]

square with optimal adjoined isosceles triangle

\[
\approx 15.07344594
\]

square with single optimally rounded corner (radius is one half the side length)

\[
12 + \pi \approx 15.14159265
\]

square with four adjoined semicircles

\[
\frac{8\pi^2}{2 + \pi} \approx 15.35649370
\]

11 pointed star

\[
\approx 15.35751730
\]

equilateral triangle with adjoined semicircle

\[
\frac{2(4 + \pi)^2}{\pi + 2\sqrt{3}} \approx 15.44193344
\]

square

\[
16
\]

circular sector, two radians (optimal circular sector)

\[
16
\]
10 pointed star $\approx 16.06491327$

circular sector, $3\pi/4$ radians $\frac{(8 + 3\pi)^2}{6\pi} \approx 16.10769443$

quarter circular sector $\frac{4(2 + \frac{\pi}{2})^2}{\pi} \approx 16.23455083$

optimal parabolic chunk (region bounded by $y = x^2$ and $y \approx 3.127903$) $\approx 16.48263518$

parabolic chunk: region bounded by $y = x^2$ and $y = 3$ $\approx 16.48509413$

parabolic chunk: region bounded by $y = x^2$ and $y = 4$ $\approx 16.56740039$

rectangle, 3:2 sides $\frac{50}{3} = 16.6$

regular pentagon with adjoined semicircles $\frac{50\pi^2}{5\pi + 2\sqrt{25} + 10\sqrt{5}} \approx 16.74415928$
semicircle augmented by two semicircles, I
\[ \frac{16}{3} \pi \approx 16.75516081 \]

parabolic chunk: region bounded by \( y = x^2 \) and \( y = 2 \)
\[ \approx 16.76938845 \]

parabolic chunk: region bounded by \( y = x^2 \) and \( y = 5 \)
\[ \approx 16.79033103 \]

semicircle
\[ \frac{2(2 + \pi)^2}{\pi} \approx 16.82966439 \]

golden rectangle (rectangle with side ratio 1:\( \phi \) where \( \phi = \frac{1}{2}(1 + \sqrt{5}) \))
\[ 8\phi + 4 \approx 16.944271909 \]

rectangle with side ratio 1:1.63712909...
\[ \approx 16.99181787 \]

rectangle with side ratio 1:1.63712909... with adjoined semicircles
\[ \approx 16.99181787 \]
equilateral triangle with adjoined optimal rectangle \[ 4(6 - \sqrt{3}) \approx 17.0719676 \]

9 pointed star \[ \approx 17.10465828 \]

square with adjoined equilateral triangle \[ \frac{100}{4 + \sqrt{3}} \approx 17.44576302 \]

regular hexagon with adjoined semicircles \[ \frac{12\pi^2}{\pi + 2\sqrt{3}} \approx 17.92926648 \]

rectangle, 2:1 \[ 18 \]

circular sector, one radian \[ 18 \]

circular sector, four radians \[ 18 \]

rectangle, 2:1, with one adjoined semicircle \[ \frac{2(10 + \pi)^2}{16 + \pi} \approx 18.044627800654 \]
rectangle, 2:1, with two adjoined semicircles

\[ \frac{4(4 + \pi)^2}{8 + \pi} \approx 18.31061221 \]

parabolic chunk: region bounded by \( y = x^2 \) and \( y = 10 \)

\[ \approx 18.38876821 \]

parabolic chunk: region bounded by \( y = x^2 \) and \( y = 1 \)

\[ \approx 18.43547307 \]

8 pointed star

\[ 64 - 4^{11/4} \approx 18.74516600 \]

regular septagon with adjoined semicircles

\[ \approx 18.94192830 \]

ellipse, 3:1

\[ \approx 18.95220996 \]

three-quarter circular sector

\[ \frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068 \]

quarter circular sector with half-radius quarter circular sector removed

\[ \frac{(4 + 3\pi)^2}{3\pi} \approx 19.12243068 \]
square with an adjoined semicircle on each side’s middle third
\[ \frac{8(\pi + 4)^2}{18 + \pi} \approx 19.29933907 \]

isosceles right triangle with optimal adjoined rectangle (1 : \(\frac{\sqrt{2}}{2}\))
\[ 8 + 8\sqrt{2} \approx 19.31370849 \]

isosceles right triangle with adjoined square
\[ \frac{2(4 + \sqrt{2})^2}{3} \approx 19.54247233 \]

square with square adjoined to middle-third of one side
\[ \frac{98}{5} = 19.6 \]

circular sector, \(\pi/4\) radians
\[ \frac{(8 + \pi)^2}{2\pi} \approx 19.756712684 \]

regular octagon with adjoined semicircles
\[ \approx 19.81346029 \]

parabolic chunk: region bounded by \(y = x^2\) and \(y = 15\)
\[ \approx 20.00749094 \]
square with two equilateral triangles adjoined to one side
\[\frac{200}{8 + \sqrt{3}} \approx 20.55065309\]

regular nonagon with adjoined semicircles
\[\approx 20.569895\]

equilateral triangle
\[\frac{36}{\sqrt{3}} \approx 20.78460969\]

regular decagon with adjoined semicircles
\[\approx 21.231897\]

two step unit staircase
\[\frac{64}{3} = 21.3\]

square with quarter-sized squares removed from each corner (aka order-2 Aztec diamond)
\[\frac{64}{3} = 21.3\]

rectangle, 3:1
\[\frac{64}{3} = 21.3\]

circular sector, 2/3 radians
\[\frac{64}{3} = 21.3\]
circular sector, 6 radians

\[
\frac{64}{3} = 21.3
\]

parabolic chunk: region bounded by \( y = x^2 \) and \( y = 20 \)

\[\approx 21.50079173\]

7 pointed star

\[\approx 21.62680012\]

rectangle, 3:1, with one adjoined semicircle

\[\frac{2(14 + \pi)^2}{24 + \pi} \approx 21.65194964\]

regular 11-gon with adjoined semicircles

\[\approx 21.81573534\]

square with three equilateral triangles adjoined to one side

\[
\frac{300}{12 + \sqrt{3}} \approx 21.84670041
\]

rectangle, 3:1, with two adjoined semicircles

\[\frac{4(6 + \pi)^2}{12 + \pi} \approx 22.076598718\]

regular 12-gon with adjoined semicircles

\[\approx 22.33427625\]
optimal “45 degree” trapezoid (base= 1, height= \(\frac{1}{\sqrt{2}+1}\)) (aka optimal truncated isosceles right triangle aka optimal rectangle with two adjoined isosceles right triangles)

\[16\sqrt{2} \approx 22.62741699796\]

regular 13-gon with adjoined semicircles

\[\approx 22.79777267\]

isosceles triangle, 2:1 sides, with adjoined semicircle

\[
\frac{2(8 + \pi)^2}{\pi + 2\sqrt{15}} \approx 22.80310635
\]

regular 14-gon with adjoined semicircles

\[\approx 23.21447316\]

square with squares adjoined to middle third of two sides

\[
\frac{256}{11} = 23.27
\]

isosceles right triangle

\[2(2 + \sqrt{2})^2 \approx 23.31370849\]

square with two adjoined isosceles right triangles

\[2(2 + \sqrt{2})^2 \approx 23.31370849\]
square with four adjoined equilateral triangles

\[ \frac{64}{1 + \sqrt{3}} \approx 23.42562584 \]

regular 15-gon with adjoined semicircles

\[ \approx 23.59107664 \]

equilateral triangle with three adjoined squares

\[ \frac{324}{12 + \sqrt{3}} \approx 23.59443644 \]

square with an adjoined equilateral triangles on each side’s middle third

\[ \frac{256}{9 + \sqrt{3}} \approx 23.85378196 \]

regular 16-gon with adjoined semicircles

\[ \approx 23.93307364 \]

three-step unit staircase

24

order-3 Aztec diamond

24
parabolic chunk: region bounded by $y = x^2$ and $y = 30$ \[ \approx 24.15608124 \]

regular 17-gon with adjoined semicircles \[ \approx 24.245034 \]

rectangle, 4:1 \[ 25 \]

circular sector, 1/2 radian \[ 25 \]

regular 20-gon with adjoined semicircles \[ \approx 25.03530855 \]

square with two equilateral triangles adjoined to two sides \[ \frac{144}{4 + \sqrt{3}} \approx 25.1218987469293 \]

semicircle augmented with two semicircles, II \[ 8\pi \approx 25.13274122 \]

rectangle, 4:1, with one adjoined semicircle \[ \frac{2(18 + \pi)^2}{32 + \pi} \approx 25.43805821 \]
four-step unit staircase

order-4 Aztec diamond

regular pentagon with adjoined equilateral triangles

isosceles triangle, 2:1 sides

right triangle, 30-60-90

rectangle, 4:1, with two adjoined semicircles

parabolic chunk: region bounded by $y = x^2$ and $y = 40$

order-5 Aztec diamond
regular 30-gon with adjoined semicircles \[\approx 26.71031462\]

square with squares adjoined to the middle thirds of three sides \[= 27\]

square with three adjoined right isosceles triangles \[\frac{68 + 48\sqrt{2}}{5} \approx 27.17645019\]

right triangle, 2:1 legs \[(3 + \sqrt{5})^2 = 14 + 6\sqrt{5} \approx 27.41640786\]

order-6 Aztec diamond \[\frac{192}{7} = 27.128571\]

square with one-third diameter semicircles removed from each side \[\frac{2(8 + 2\pi)^2}{18 - \pi} \approx 27.46046434\]

semicircle with half-diameter semicircle removed \[\frac{2(3\pi + 2)^2}{3\pi} \approx 27.69838228\]

6 pointed star (Koch curve, stage 2; regular hexagon with adjoined equilateral triangles) \[\sqrt{768} \approx 27.71281292\]
order-7 Aztec diamond

regular 50-gon with adjoined semicircles

parabolic chunk: region bounded by $y = x^2$ and $y = 50$

5-square cross

rectangle, 5:1

circular sector, $2/5$ radian

regular septagon with adjoined equilateral triangles

rectangle, 5:1, with two adjoined semicircles

\[
\frac{4(10 + \pi)^2}{20 + \pi} \approx 29.851265651086
\]
13-square diamond

\[ \frac{400}{13} \approx 30.76923076 \]

regular octagon with adjoined equilateral triangles

\[ \approx 30.87116222 \]

5-square cross with four adjoined semicircles

\[ \frac{8(4 + \pi)^2}{\pi + 10} \approx 31.04789318 \]

square with four adjoined isosceles right triangles

\[ 16 + \frac{32}{3} \sqrt{2} \approx 31.08494466 \]

25-square diamond

\[ \frac{784}{25} = 31.36 \]

regular nonagon with adjoined equilateral triangles

\[ \approx 32.14624235 \]

isosceles triangle, 20° apex angle

\[ \approx 32.21915602 \]

rectangle, 6:1

\[ \frac{98}{3} = 32.6 \]
circular sector, $1/3$ radian

\[ \frac{98}{3} = 32.6 \]

isosceles triangle, 3:1 sides

\[ \frac{196}{\sqrt{35}} \approx 33.13004679 \]

regular decagon with adjoined equilateral triangles

\[ \approx 33.26587053 \]

rectangle, 6:1, with two adjoined semicircles

\[ \frac{(24 + 2\pi)^2}{24 + \pi} \approx 33.78841190 \]

square with two adjoined equilateral triangles on three sides

\[ \frac{450}{8 + 3\sqrt{3}} \approx 34.10084891 \]

regular 11-gon with adjoined equilateral triangles

\[ \approx 34.2563207 \]

square with two adjoined equilateral triangles on each side

\[ \frac{128}{2 + \sqrt{3}} \approx 34.29749663 \]

regular 12-gon with adjoined equilateral triangles

\[ \approx 35.13843876 \]
region bounded by $y = x^2, y = 0, x = 1$ \[ \approx 36.30913022 \]

regular 20-gon with adjoined equilateral triangles \[ \approx 35.13843876 \]

rectangle, 7:1 \[ \frac{256}{7} \approx 36.57142857 \]

circular sector, $2/7$ radian \[ \frac{256}{7} \approx 36.57142857 \]

three-quarter circular sector with half-radius three-quarter circular sector removed \[ \frac{(9\pi + 4)^2}{9\pi} \approx 36.84021812 \]

parabolic chunk: region bounded by $y = x^2$ and $y = 100$ \[ \approx 36.99450714 \]

regular hexagon with adjoined squares \[ \approx 37.682848 \]

rectangle, 8:1 \[ \frac{81}{2} = 40.5 \]
circular sector, $1/4$ radian

\[ \frac{81}{2} = 40.5 \]

square with three equilateral triangles adjoined to each side

\[ \frac{192}{3 + \sqrt{3}} = 40.57437416 \]

isosceles triangle, 4:1 sides

\[ \frac{324}{\sqrt{63}} \approx 40.82016308 \]

regular septagon with adjoined squares

\[ \approx 41.471095 \]

regular 30-gon with adjoined equilateral triangles

\[ \approx 42.68026645 \]

Koch curve, stage 3

\[ \frac{384}{5\sqrt{3}} \approx 44.34050067 \]

9-square cross

\[ \frac{400}{9} = 44.\bar{4} \]

rectangle, 9:1

\[ \frac{400}{9} = 44.\bar{4} \]
circular sector, $\frac{2}{9}$ radian

\[
\frac{400}{9} = 44.\bar{4}
\]

regular octagon with adjoined squares

\[\approx 44.900282\]

regular 50-gon with adjoined equilateral triangles

\[\approx 45.38596197\]

5 pointed star

\[20\sqrt{10} - 2\sqrt{5} \approx 47.02282018\]

regular nonagon with adjoined squares

\[\approx 48.017945\]

rectangle, 10:1

\[\frac{242}{5} = 48.4\]

circular sector, $\frac{1}{5}$ radian

\[\frac{242}{5} = 48.4\]

semicircle augmented by two semicircles, III (arbelos)

\[16\pi \approx 50.26548245\]
region bounded by $y = x^3$, $y = 0$ and $x = 1$ \[\approx 50.34940281\]

regular decagon with adjoined squares \[\approx 50.864099\]

rectangle, 11:1 \[\frac{576}{11} = 52.36\]

circular sector, 2/11 radian \[\frac{576}{11} = 52.36\]

regular 11-gon with adjoined squares \[\approx 53.472417\]

isosceles triangle, 10° apex angle \[\approx 54.45067722\]

arbelos, one-third and two-thirds arcs \[18\pi \approx 56.54866776\]

arbelos, one-quarter and three-quarter arcs \[\frac{64\pi}{3} \approx 67.02064328\]
regular 20-gon with adjoined squares  \[ \approx 69.80970978 \]

Koch curve, stage 4  \[ \frac{6144}{47\sqrt{3}} \approx 75.47319263 \]

arbelos, one-fifth and four-fifths arcs  \[ 25\pi \approx 78.53981634 \]

regular 30-gon with adjoined squares  \[ \approx 79.91496779 \]

region bounded by \( y = x^4, y = 0 \) and \( x = 1 \)  \[ \approx 80.73353354 \]

arbelos, one-sixth and five-sixths arcs  \[ \frac{144\pi}{5} \approx 90.47786842 \]

isosceles triangle, 5° apex angle  \[ \approx 99.97197122 \]

Koch curve, stage 5  \[ \frac{98304}{431\sqrt{3}} \approx 131.68408553 \]
arbelos, one-tenth and nine-tenths arcs

\[
\frac{400\pi}{9} \approx 139.62634016
\]
General formulas

Rectangles
For a rectangle with aspect ratio $m$ (i.e., the ratio of non-equal length sides is $m : 1$), the ratio is

\[ \frac{(2m + 2)^2}{m} = 4m + 8 + \frac{4}{m}. \]

Right triangles
For a right triangle with non-right angle $\theta$, the ratio is

\[ \frac{2(1 + \sin \theta + \cos \theta)^2}{\sin \theta \cos \theta}. \]

Isosceles triangles
For an isosceles triangle in which the sides with the same length are $l$ times the length of the other side, the ratio is

\[ \frac{4(2l + 1)^2}{\sqrt{4l^2 - 1}}. \]

For an isosceles triangle with an “apex” angle of $\theta$, the ratio is

\[ 4 \tan \frac{\theta}{2} \left( 1 + \csc \frac{\theta}{2} \right)^2. \]

Rectangle with adjoined semicircles
For a rectangle with aspect ratio $m$ with semicircles adjoined to the ”1” sides, the ratio is

\[ \frac{(4m + 2\pi)^2}{4m + \pi}. \]

For a value of $m \approx 1.637129085772\ldots$, this is equal to the ratio for the $m : 1$ rectangle. If $m > 1.637129085772\ldots$, adjoining the semicircles decreases the ratio, while for smaller $m$, adjoining semicircles increases the ratio.

Regular polygons
For a regular polygon with $n$ sides, the ratio is

\[ 4n \tan \frac{\pi}{n}. \]

Circular sectors
For a circular sector with angle $\theta$, the ratio is

\[ \frac{2(2 + \theta)^2}{\theta}. \]

This has a minimum of 16 at $\theta = 2$.

Polygonal Stars
For an $n$-sided polygonal star with equal side lengths, like the one shown, with inner radius $r$ and outer radius $R$, the area is $\frac{1}{2}nrR \sin \frac{2\pi}{n}$ and the perimeter is $n \sqrt{r^2 + R^2 - 2rR \cos \frac{2\pi}{n}}$ so the isoperimetric ratio is

\[ \frac{2n(r^2 + R^2 - 2rR \cos \frac{2\pi}{n})}{rR \sin \frac{2\pi}{n}}. \]

A special case of this are stars based on a regular polygon with $m \geq 5$ sides by extending the sides of the polygon until they intersect. In this case, we have $n = 2m$ and $R = r(\cos \frac{\pi}{m} + \sin \frac{\pi}{m} \tan \frac{2\pi}{m})$. In the table, such stars are called $n$-pointed stars.
Regular polygons with adjoined equilateral triangles
If we adjoin equilateral triangles to the sides of a regular \( n \)-gon, the resulting figure has an isoperimetric ratio of

\[
\frac{16n}{\sqrt{3} + \cot \frac{\pi}{n}}.
\]

As \( n \) tends to infinity, this approaches \( 16\pi = 50.265482 \ldots \) from below.

Regular polygons with adjoined squares
If we adjoin squares to the sides of a regular \( n \)-gon, the resulting figure has an isoperimetric ratio of

\[
\frac{36n}{4 + \cot \frac{\pi}{n}}.
\]

As \( n \) tends to infinity, this approaches \( 36\pi = 113.097 \ldots \) from below.

Regular polygons with adjoined semicircles
If we adjoin semicircles to the sides of a regular \( n \)-gon, the resulting figure has an isoperimetric ratio of

\[
\frac{\pi^2 n}{\frac{\pi}{2} + \cot \frac{\pi}{n}}.
\]

As \( n \) tends to infinity, this approaches \( \pi^3 \approx 31.0063 \ldots \) from below.

\( n \)-square diamond/cross
The ratio is

\[
\frac{(8m + 12)^2}{2m^2 + 6m + 5}
\]

where \( m \) is the order of the diamond (\( m = 0 \) is the 5-square cross, \( m = 1 \) is the 13-square diamond, etc.)
This approaches 32 from below as \( m \) tends to infinity.

Koch curve
The ratio is

\[
\frac{180 \left(\frac{4}{3}\right)^{2n}}{\sqrt{3}(8 - 3 \left(\frac{4}{3}\right)^n)}
\]

for the \( n \)-th iteration (i.e., \( n = 0 \) is an equilateral triangle, \( n = 1 \) is a six-pointed star, etc.)

Arbelos

The area is \( \frac{\pi}{4} q(1 - q) \) and the perimeter is \( \pi \), so the ratio is

\[
\frac{4 \pi}{q(1 - q)}.
\]

Parabolic chunk
For regions bounded by a parabola and a line perpendicular to the parabola’s axis, the ratio is

\[
\left( \frac{2 + \sqrt{1 + 4a^2} + \frac{1}{2a} \log(2a + \sqrt{1 + 4a^2})}{\frac{4}{3}a} \right)^2.
\]
where the parabola is \( y = ax^2 \) and the bounding line is \( y = a \) (we can express all parabolic chunks in this form; note, for instance, that the shape bounded by \( y = x^2 \) and \( y = b \) is the same as the shape bounded by \( y = \sqrt{bx^2} \) and \( y = \sqrt{b} \)).

**Aztec diamond**

For an \( n \)-th order Aztec diamond, which looks like four copies of a set of \( n \) steps (of unit width and height) stuck together, the area is \( 2n(n + 1) \) and the perimeter is \( 8n \), so the ratio is

\[
\frac{P^2}{A} = \frac{64n^2}{2n(n + 1)} = \frac{32n}{n + 1}.
\]

Incidentally, this is the same ratio as for a set of \( n \) unit steps themselves, as the perimeter is \( 4n \) and the area is \( \frac{1}{2}n(n + 1) \) which yields

\[
\frac{P^2}{A} = \frac{(4n)^2}{\frac{1}{2}n(n + 1)} = \frac{32n}{n + 1}.
\]

This is because the “stair” part and the non-stair part have the same length, so putting four of them together exactly doubles the perimeter while quadrupling the area (unlike what happens when you put, say, four quarter-circles together to make a circle).