MATH 300 D, Autumn 2014
Midterm II Study Problems

1. Prove that, for all $x \in \mathbb{Z}$, if $x^2 - 1$ is divisible by 8, then x is odd.

2. Prove or give a counterexample for each of the following statements.

   (a) For all real numbers x and y, $|x + y| = |x| + |y|$.
   (b) For all real numbers x and y, $|xy| = |x||y|$.
   (c) There is a positive integer M such that, for every positive integer $n > M$, $\frac{1}{n} < 0.002$.
   (d) For all integers a and b, if $a|b$ and $b|a$, then $a = b$ or $a = -b$.
   (e) For all integers m and n, if $n + m$ is odd, then $n \neq m$.

3. (a) Let x be an integer. Prove that if $\sqrt{2x}$ is an integer, then x is even.
   (b) Is the converse of the statement you proved in (a) true? Prove it or give a counterexample.
   (c) What can you conclude about $\sqrt{2x}$ if x is odd?

4. (a) Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\bigcup \mathcal{F} \subseteq B$ then $\mathcal{F} \subseteq \mathcal{P}(B)$.
   (b) Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets. Suppose every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Then $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$.

5. Define a relation T on the set $\mathbb{R}$ of real numbers by
$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1\}.$$
   Is T an equivalence relation? (Justify your answer, of course.)

6. Define a relation R on $\mathbb{Z}$ by
$$(x, y) \in R \Leftrightarrow x - y \text{ is even.}$$
   Determine whether or not R is reflexive, symmetric and transitive. Is R an equivalence relation? If R is an equivalence relation, describe its equivalence classes.

7. Define a relation R on $\mathbb{Z}$ by
$$(x, y) \in R \Leftrightarrow xy \equiv 0 \pmod 4.$$
   Determine whether or not R is reflexive, symmetric and transitive. Is R an equivalence relation? If R is an equivalence relation, describe its equivalence classes.

8. Let A be the set of all real functions $f : \mathbb{R} \to \mathbb{R}$. Define a relation R on A by:
$$(f, g) \in R \Leftrightarrow \text{there exists a real constant } k \text{ such that } f(x) = g(x) + k \text{ for all } x \in \mathbb{R}.$$
   Prove that R is an equivalence relation.

9. Define a relation R on $\mathbb{R}$ by:
$$(x, y) \in R \Leftrightarrow |x - y| < 1$$
   Prove that R is not an equivalence relation.

10. Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

11. Let $m \in \mathbb{Z}$ and suppose $m > 1$. Suppose $a, b, c \in \mathbb{Z}$.
    Prove that if $a \equiv b \pmod m$, then $ac \equiv bc \pmod m$.

12. Prove that if n is an integer, then $n^2 \equiv 0, 1, \text{or } 4 \pmod 8$.